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Long Baseline Underwater Positioning with Fusion of Saved and Current Measurements and Ambiguity Resolution. Part I. Mathematical Formulation

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Abstract: We report a recursive positioning algorithm for an autonomous underwater vehicle (AUV) based on measurements of ranges to acoustic beacons, water speed log and heading indicator data. Two types of desynchronization between the beacon and AUV clocks are considered: random and unknown. The algorithm starts without using AUV a priori coordinates when simultaneous measurements from minimum two or three beacons (depending on the desynchronization type) are first obtained. The newly coming measurements and those saved before the algorithm start are processed in forward and backward time in the same filter. If AUV coordinate estimates are ambiguous, two filters are implemented, which process the same data with different measurement linearization points. Ambiguity is resolved based on the ratio of a posteriori probabilities of hypotheses on AUV position. This ratio is calculated using the filters' outputs.

Keywords: autonomous underwater vehicle, long baseline navigation, range and range difference measurements, dead-reckoning, Kalman filter, ambiguity, a posteriori probability.

INTRODUCTION

Navigation provision of autonomous underwater vehicles (AUVs) generally relies on autonomous devices such as a log, heading indicator, and inertial navigation system providing the dead-reckoning, and correction aids such as a hydroacoustic system measuring the ranges to the reference beacons. The correction efficiency depends on the number of AUV on board receivers and the beacons, their geometrical arrangement, and mathematical approaches applied in measurement processing [1–19].

A large part of modern publications on this topic are devoted to long baseline (LBL) navigation method [8, 11, 17–19]. Implementation of this method, despite its apparent simplicity, can be nontrivial under certain limitations and additional conditions. In this paper the LBL method is utilized in non-standard conditions: insufficient number of beacons for unambiguous navigation solution, their unfavorable arrangement degrading the accuracy, and no a priori AUV coordinates. The beacons' coordinates are considered to be known, the desynchronization between the beacon and AUV clocks can be random (with known characteristics) or unknown. The range measurement is determined as a product of the measured one-way travel time from the beacon to AUV and approximate estimate of sound speed in water. The problem is solved by the series of time-different measurements using log and heading indicator data. Deterministic cross-range method and its subtypes using sample time-different measurements [20] can be applied. A more efficient solution can be obtained by using all the available measurements and taking account of the statistical features of their errors, i.e., within the stochastic approach. Note that we are searching for a solution applicable in AUV onboard hardware in real time, where it is rational to employ suboptimal stochastic estimators of low computational complexity.

If a priori AUV position is sufficiently accurate for measurement linearization, the LBL method can be implemented using the extended or iterated Kalman filter (KF) [4, 21, 22]. However, in the case considered no a priori coordinates are available, and these methods cannot be applied on their own. The proposed algorithm relies on methods for processing the linearized measurements as the simplest ones, which are supplemented with preparation procedures to provide proper performance. The algorithm starts when the quantity of simultaneous measurements is sufficient to obtain the navigation solution, though ambiguous, in the form of two possible AUV positions.

For initial linearization we can use the approximate coordinates obtained by analytical solution of equations constructed from the measurements with no consideration for their errors, as, for example, in [23–26]. This approach is utilized in [13, 27] to process range hydroacoustic measurements. In the algorithm proposed below, the coordinates for linearization are determined with account of the features of measurement errors, which makes them closer to final estimates and enhances the linearization accuracy.

The measurements arriving before the algorithm start are assumed to have been saved for further processing. The solution processing all the saved measurements at once by fixed-point smoothing [28] has been studied in [29]. Here, the fixed point is the time when the algorithm starts, however, processing a large number of saved measurements can turn out too long. It is expedient to process the saved measurements by parts, gradually including them in the solution in the reverse order so that to avoid delays in processing the measurements arriving after the algorithm start. In [27] the saved measurements are processed recursively. The saved measurements are processed in backward time, and current measurements are processed in forward time with two KFs. Their outputs are fused by the dummy measurements method [30] used to correct KF outputs so that they agree with other a priori data. In this problem, the data from KF estimating the initial and current state vectors in forward time are updated using dummy measurements based on outputs from KF estimating the initial and one of the earlier state vectors in backward time. Here we present an easier programming solution where the saved and current measurements are processed together in a common KF.

In case of ambiguous solution, two KFs are running that use different linearization points. Based on their outputs, the ratio of a posteriori probabilities of hypotheses on AUV position is determined and further analyzed to select the true one. This solution relies on multiple model filtering theory used in some navigation applications [21, 31], in particular in AUV single-beacon navigation with no a priori coordinates [32]. Note a multi-KF algorithm or a polygaussian filter ([21], pp. 87–89) using different linearization points as in this paper. However, this algorithm assumes using a priori information on state vector for determining the initial linearization points, which is impossible in the considered conditions.

The key features of the presented algorithm is obtaining the initial navigation solution without a priori AUV coordinates using a minimal number of beacons (for navigation solutions in the form of certain points) and recursive processing of saved and current measurements with a common KF. The ambiguity is resolved, as mentioned before, based on a known multiple model filtering approach. Here it is adapted to coprocessing the current and saved measurements in forward and backward times and protected from the errors caused by the limitation of bit grid.

This algorithm with some modifications can be applied to positioning of other vehicles with dead-reckoning sensors and systems measuring ranges or range differences to the beacons or point landmarks, the number and arrangement of which does not provide the full navigation coverage. In indoor environments, these could be pseudolites [33], Ultra-Wideband, Bluetooth, Wi-Fi sources, etc. [34].

This paper develops and refines the studies from [35, 36], where the algorithm is presented in a brief form ([35] concerns random desynchronization only). The first part of the paper includes four sections. Section 1 formulates the mathematical problem statement. Section 2 details the processing algorithm for measurements at the beginning of solution. Section 3 reports and discusses the fusion of current and saved measurements with one KF. Section 4 focuses on ambiguous solutions. The second part of the paper will deal with the algorithm runtime estimates, results from simulation and real data postprocessing, which confirm the algorithm efficiency.

1. AUV POSITIONING PROBLEM STATEMENT

At discrete time moments t_i, i is an integer index, the measured ranges to beacons are available on AUV:

$$Y_i^j = \mathcal{T}_i^j \hat{c}_0 = \sqrt{(x_i^j - x_i)^2 + (y_i^j - y_i)^2 + (z_i^j - z_i)^2} + \Delta c T_i^j + \delta_i + v_i^j, \quad (1)$$

where $j = 1, \dots, n_i$ is the beacon number, x_i, y_i are the AUV unknown coordinates in the local Cartesian frame with geographically oriented axes (no a priori data are available on them), x_i^j, y_i^j are the known horizontal coordinates of beacons, z_i^j, z_i are the known depths of the beacons' installation and AUV position, $\hat{c}_0$, Δc are a priori estimate of sound speed in water and its error, being a random constant with standard deviation (SD) $\sigma_{\Delta c}$, $\mathcal{T}_i^j$, T_i^j are the measured and computed one-way signal travel time, v_i^j is a white noise error with SD σ_v noncorrelated for different beacons, δ_i is the error common for all the beacons caused by desynchronization of beacon and AUV clocks. For brevity, δ_i further is referred to as the desynchronization, and numbers i of discrete time moments denote the moments t_i. The desynchronization δ_i is considered as a random error or as an unknown value. Random δ_i is given by

$$\delta_i = b + e_i,$$

where b is a random bias with SD σ_b, e_i is the white noise with SD σ_e. Differences between the beacons' time scales and variations of the sound speed are assumed negligibly small or can be partly referred to v_i^j.

The AUV is assumed to be equipped with a dual axis electromagnetic speed log and a heading indicator, for example, a magnetic compass. The log generates the longitudinal and transverse (positive to starboard) velocity components $\tilde{V}^{x*}, \tilde{V}^{y*}$ with white noise instrumental errors with SD averaged over a unit interval $\sigma_{\Delta V}$. The heading indicator redaings $\tilde{K}$ have an error ΔK – a stationary first-order Markov process with SD $\sigma_{\Delta K}$ and correlation interval $\tau_{\Delta K}$. We also know the approximate geographical components of the current speed $\tilde{U}^x$, $\tilde{U}^y$ with the errors $\Delta U^x, \Delta U^y$ being stationary first-order Markov process with SD $\sigma_{\Delta U}$ and correlation interval $\tau_{\Delta U}$. Suppose that $\tilde{K}$, $\tilde{V}^{x*}, \tilde{V}^{y*}$, $\tilde{U}^x$, $\tilde{U}^y$ are recalculated to the acoustic measurement arrival times using averaging, interpolation, or extrapolation if required. Using the dead-reckoning method, the recursive equations for AUV horizontal coordinates are written as follows:

$$\begin{aligned} x_i &= x_{i-1} + (\tilde{V}_{i-1}^y \Delta K_{i-1} + \tilde{V}_{i-1}^x + \tilde{U}_{i-1}^x + \Delta U_{i-1}^x)\Delta t_i + w_{i-1}^x, \\ y_i &= y_{i-1} + (-\tilde{V}_{i-1}^x \Delta K_{i-1} + \tilde{V}_{i-1}^y + \tilde{U}_{i-1}^y + \Delta U_{i-1}^y)\Delta t_i + w_{i-1}^y. \end{aligned} \quad (2)$$

where $\Delta t_i = t_i - t_{i-1}$, $\tilde{V}_{i-1}^x = \tilde{V}_{i-1}^{x*} \sin \tilde{K}_{i-1} + \tilde{V}_{i-1}^{y*} \cos \tilde{K}_{i-1}$, $\tilde{V}_{i-1}^y = \tilde{V}_{i-1}^{x*} \cos \tilde{K}_{i-1} - \tilde{V}_{i-1}^{y*} \sin \tilde{K}_{i-1}$, w^x, w^y are the generating white noises with SD $\sigma_{\Delta V} \cdot \Delta t_i$.

All the random values occurring in the problem are assumed to have zero-mean Gaussian probability distribution.

Introduce the index k denoting the type of desynchronization δ_i: $k = 1$ – random δ_i, $k = 2$ – unknown δ_i. Determine the time moment $i = 0$, when the algorithm is started. Suppose that it is the time when simultaneous measurements from more than k beacons are first input to the algorithm, i.e., measurements from two and more beacons with random desynchronization, and measurements from three and more beacons with unknown desynchronization. Let $i = -N^k$ ($N^k \geq 0$) denote the time when measurements from k beacons are first received. Thus, we have

$$n_0 \geq k + 1, \; n_i < k \text{ with } i < -N^k,$$

$$\text{if } N^k \neq 0, \text{ then } n_{-N^k} = k, \; n_i \leq k \text{ with } -N^k < i \leq -1.$$

With unknown δ_i ($k = 2$), range difference measurements (further, sometimes called difference measurements) $\Delta Y_i^j = Y_i^{j+1} - Y_i^1$, $j = 1, \ldots, n_i - 1$ can be used in the solution, where δ_i is excluded, and the number of measurements is $n_i - 1$, but range measurements Y_i^j can be directly used too with the observed conditions for n_i. If $k = 2$ and only one measurement Y_i^1 is available at the i-th time, it is ignored. The number of the used beacons with $i > 0$ for both types of δ_i is not specified. Before $i = 0$ – the algorithm starting time, the measurements Y_i^j, log data $\tilde{V}_i^{x*}, \tilde{V}_i^{y*}$, and heading data $\tilde{K}_i$ together with the moments t_i are saved for further use. Figure 1 explains the i scales for both types of δ_i.

It is needed to determine AUV horizontal coordinates x_i, y_i at times $i \geq 0$ by all available acoustic measurements (1), including those saved before $i = 0$ using (2) and stochastic description of data errors. The problem is reduced to estimating the state vector

$$X_i = (x_i, y_i, \Delta c, b, \Delta K_i, \Delta U_i^x, \Delta U_i^y)^{\mathrm{T}} \text{ with } k = 1,$$

$$\text{or } X_i = (x_i, y_i, \Delta c, \Delta K_i, \Delta U_i^x, \Delta U_i^y)^{\mathrm{T}} \text{ with } k = 2,$$

by measurements $Y_{-N^k}, \ldots, Y_i$, where the vectors $Y_i = (Y_i^1, \ldots, Y_i^{n_i})^{\mathrm{T}}$ are formed under the condition $n_i \geq k$.

Below we provide some explanations regarding the problem statement.

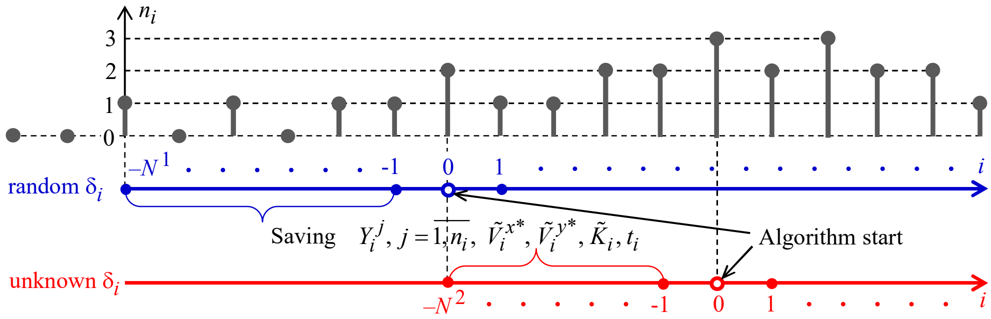

Fig. 1. Diagram showing the number of simultaneously observed beacons with i scales for two types of desynchronization δ.

It is supposed to use the pinger beacons regularly emitting the signals. If the beacons are installed on the surface buoys, they generate the signals constantly without request from the AUV and replenish the energy supply from the solar batteries. If bottom beacons are applied, their power supply is low, they generate signals on AUV request for a finite time. To send a request to the bottom beacons, AUV should know that it is within the beacon coverage area, therefore, in the case of bottom beacons we do not speak about complete absence of a priori information on AUV position.

Synchronizing the time scales of surface buoy beacons generally does not constitute a problem, because these buoys can be equipped with receivers of signals of global navigation satellite systems. The time scale of each beacon can be accurately synchronized with coordinated universal time UTC or clock of one of the satellite system with 1 PPS signal. The bottom beacons are synchronized by exchanging signals [37]. Obviously, the buoys in open water or on ice can be applied if their drift allows performing AUV mission over a limited time interval. The bottom beacons should be located at ranges providing the signal exchange.

The measurement sampling interval $\Delta t_i = t_i - t_{i-1}$ can be varying, but it is close to a known period $\Delta\bar{t}$ of beacon signal generation. If no acoustic measurement have been received by AUV before $t_{i-1} + \Delta t_{\max}$, where $\Delta t_{\max} \gtrsim \Delta\bar{t}$ is the preset value, $t_i = t_{i-1} + \Delta\bar{t}$ is accepted, and the coordinates are dead-reckoned at that time.

The coordinates are directly included in the state vector, which is unlike the classical scenarios of navigation data fusion, where the differences between the readings of autonomous navigation systems (inertial or dead-reckoning) and aiding measurements [4, 21] are applied. In these problems, the state vector includes the errors of autonomous systems including the coordinate errors but not the coordinates. In the considered problem, the coordinate dead-reckoning begins at the algorithm start $i = 0$. The dead-reckoning procedure is embedded in the position estimation algorithm.

Unknown desynchronization δ_i can be treated as a white noise with infinite SD. However, unknown δ_i is considered as a separate case, because numerical implementation of the solution with white noise δ_i with finite but very large SD is problematic.

The algorithm can be actually started when only one range or range difference measurement is available using single-beacon range navigation methods as in [32, 38–43]. However, these methods can turn quite labor-consuming under no a priori AUV position. Moreover, to obtain an unambiguous solution in these conditions AUV path should greatly differ from the rectilinear one, which can contradict AUV mission.

2. MEASUREMENT PROCESSING AT THE ALGORITHM START ($i = 0$)

To apply a computationally simple measurement processing algorithm based on linearization of measured parameters, it is essential to determine the initial linearization point that would ensure the algorithm convergence. Since AUV a priori position is not available according to the problem conditions, this point can be obtained only from the measurements. Determining the linearization point and estimating the state vector at the beginning of the algorithm is discussed in this section.

In this problem we encounter a serious difficulty, i.e., possible ambiguity of AUV position. Figure 2 shows the beacon configurations, simultaneous measurements from which provide unambiguous (one AUV image) or ambiguous (two AUV images) solutions. The upper row of pictures demonstrates the cases with random desynchronization δ of beacon and AUV clocks, where range measurements are directly used in the solution, with circular lines of position. The lower row of pictures shows the cases with unknown desynchronization with range difference measurements, in which desynchronization effect is excluded. Here the lines of position are the hyperbolas, though the initial solution with unknown desynchronization presented below uses the original range measurements, too.

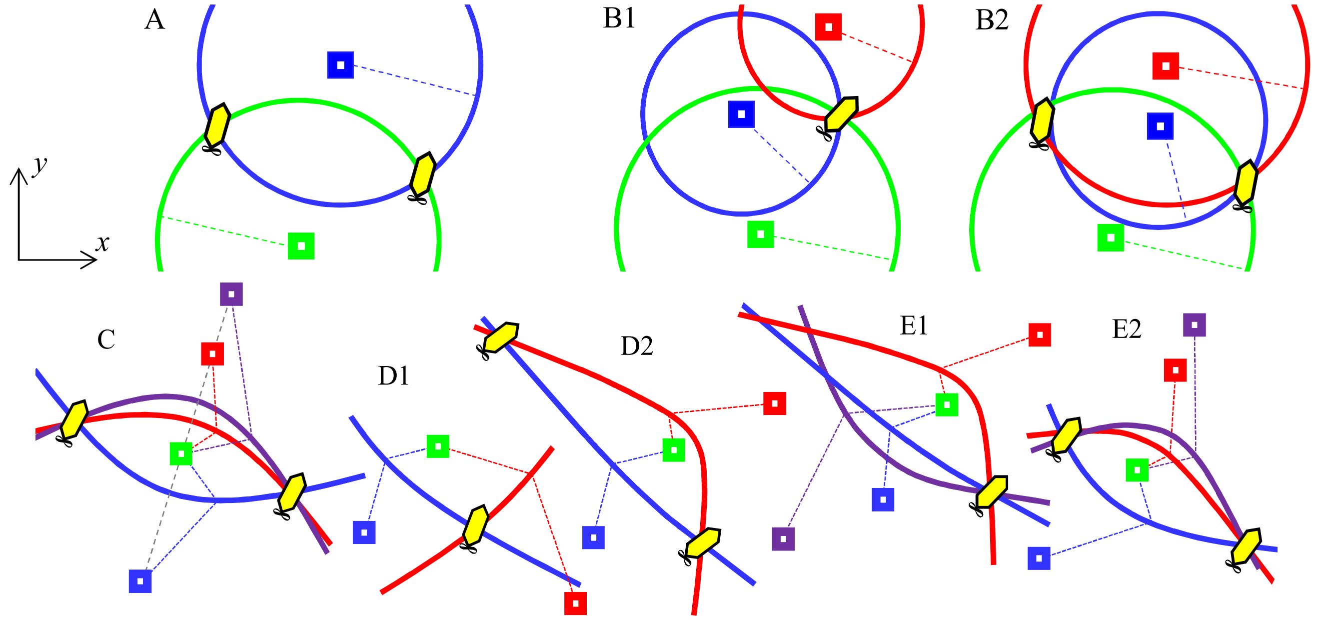

Fig. 2. Arrangements of beacons, lines of position for simultaneous range (A, B1, B2) and range difference (C, D1, D2, E1, E2) measurements, possible AUV positions

When AUV approaches a group of beacons, they are activated gradually, and n_i is unlikely to increase by more than 1 at once. However, the cases $n_0 \geq k+2$ are quite possible, for example, when the navigation algorithm is started or restarted on AUV located in a beacon rich area.

Case A: random desynchronization, $n_0 \geq 2$, $\operatorname{rank}\begin{pmatrix} x_0^1 & \cdots & x_0^{n_0} \\ y_0^1 & \cdots & y_0^{n_0} \\ 1 & \cdots & 1 \end{pmatrix} = 2$ – AUV receives signals from two beacons (A in Fig. 2) or three and more beacons, whose position projections on the horizontal plane lie on a straight line. This is a case with two equiprobable AUV positions. To simplify the computations, some of them are made not in the original frame x, y, but in the turned frame

$$\mathbf{x} = x\cos\alpha - y\sin\alpha\,,\ \mathbf{y} = x\sin\alpha + y\cos\alpha\,, \qquad (3)$$

where α is the angle between axis y and the straight line passing through the beacons' position horizontal projections, i.e., $\sin\alpha \propto x_0^2 - x_0^1$, $\cos\alpha \propto y_0^2 - y_0^1$. Axis **y** is parallel to this line, therefore, axis **x** is perpendicular to it. We'll use the same indices in AUV and beacon coordinates in **x**, **y** as they were in x, y frames. In the considered case, the algorithm consists of three steps, which will be used in other cases as well, with some modifications.

<u>Step 1</u>. The beacons' coordinates in **x**, **y** frame are determined using (3) (the beacons have the same coordinate in **x** axis). AUV coordinate $\mathbf{y}_0$ is preliminarily estimated using the measurements

$$\begin{aligned}\rho^j = {} & [(Y_0^{j+1})^2 - (Y_0^1)^2 - (\mathbf{y}_0^{j+1})^2 + (\mathbf{y}_0^1)^2 - \\ & -(z_0^{j+1} - z_0)^2 + (z_0^1 - z_0)^2]/2 \approx \\ & \approx (\mathbf{y}_0^1 - \mathbf{y}_0^{j+1})\mathbf{y}_0 + Y_0^1(\Delta c T_0^1 + \delta_0 + v_0^1) - \\ & - Y_0^{j+1}(\Delta c T_0^{j+1} + \delta_0 + v_0^{j+1}),\ j = 1, \ldots, n_0 - 1.\end{aligned} \qquad (4)$$

The expression after '≈' in (4) is written with account for the measurements (1) and equalities $(x_0^j - x_0)^2 + (y_0^j - y_0)^2 = (\mathbf{x}_0^j - \mathbf{x}_0)^2 + (\mathbf{y}_0^j - \mathbf{y}_0)^2$, $\mathbf{x}_0^1 = \cdots = \mathbf{x}_0^{n_0}$, with the omitted component

$$\begin{aligned}& [(\Delta c T_0^{j+1} + \delta_0 + v_0^{j+1})^2 - (\Delta c T_0^1 + \delta_0 + v_0^1)^2]/2 = \\ & = \delta_0[\Delta c(T_0^{j+1} - T_0^1) + v_0^{j+1} - v_0^1] + \\ & + [(\Delta c T_0^{j+1} + v_0^{j+1})^2 - (\Delta c T_0^1 + v_0^1)^2]/2.\end{aligned}$$

Note that the measurements ρ^j generated as shown in (4) before '≈' do not contain the squared desynchronization δ_0.

Estimate of $\mathbf{y}_0$ with $n_0=2$ is computed using the simplest formula

$$\tilde{\mathbf{y}}_0=\rho^1\big/(\mathbf{y}_0^1-\mathbf{y}_0^2)\,. \quad (5)$$

With $n_0\geq 3$ the estimate is computed with the least squares method

$$\tilde{\mathbf{y}}_0=\tilde{H}^{\mathrm{T}}\tilde{R}^{-1}\rho\big/(\tilde{H}^{\mathrm{T}}\tilde{R}^{-1}\tilde{H})\,, \quad (6)$$

where $\rho=(\rho^1,\ldots,\rho^{n_0-1})^{\mathrm{T}}$, $\tilde{H}=(\mathbf{y}_0^1-\mathbf{y}_0^2,\ldots,\mathbf{y}_0^1-\mathbf{y}_0^{n_0})^{\mathrm{T}}$, $\tilde{R}$ is the noise covariance matrix, whose elements are determined on the assumption $T_0{}^j=\mathcal{T}_0^j$ as

$$\begin{aligned}\tilde{R}^{j,k}=&(Y_0^1\mathcal{T}_0{}^1-Y_0^{k+1}\mathcal{T}_0{}^{k+1})(Y_0^1\mathcal{T}_0{}^1-Y_0^{j+1}\mathcal{T}_0{}^{j+1})\sigma_{\Delta c}^2+\\&+(Y_0^1-Y_0^{k+1})(Y_0^1-Y_0^{j+1})\sigma_{\delta_0}^2+\\&+[(Y_0^1)^2+\delta^{j,k}(Y_0^{j+1})^2]\sigma_v^2,\quad j,k=1,\ldots,n_0-1,\end{aligned} \quad (7)$$

where $\sigma_{\delta_0}^2=\sigma_b^2+\sigma_e^2$, $\delta^{j,k}$ is the Kronecker symbol.

A set of preliminary estimates is generated for the coordinate $\mathbf{x}_0$: $\tilde{\mathbf{x}}_0^{(l)}=\tilde{\mathbf{x}}_0^{(1)}+\delta\mathbf{x}\,l$, $l=1,\ldots,L$, where $\delta\mathbf{x}=\Delta\tilde{\mathbf{x}}/(L-1)$, $L=\lceil\Delta\tilde{\mathbf{x}}/\delta\overline{\mathbf{x}}\rceil+1$, $\delta\overline{\mathbf{x}}$ is the preset parameter, $\Delta\tilde{\mathbf{x}}$ is the length of the interval covering the preliminary estimates, $\lceil\cdot\rceil$ is rounding upward. The values $\tilde{\mathbf{x}}_0^{(1)},\Delta\tilde{\mathbf{x}}$ are set based on the maximum possible range $D_{\max}$ of beacon signal reception, i.e.,

$$\begin{aligned}\tilde{\mathbf{x}}_0^{(1)}&=\max_{j=1,\ldots,n_0}\left(\mathbf{x}_0^j-\Delta\mathbf{x}^j\right),\\ \Delta\tilde{\mathbf{x}}&=\min_{j=1,\ldots,n_0}\left(\mathbf{x}_0^j+\Delta\mathbf{x}^j\right)-\tilde{\mathbf{x}}_0^{(1)},\end{aligned} \quad (8)$$

where $\Delta\mathbf{x}^j=\sqrt{D_{\max}^2-(\mathbf{y}_0^j-\tilde{\mathbf{y}}_0)^2-(z_0^j-z_0)^2}$. Since all $\mathbf{x}_0^j$ are equal, denote them with $\mathbf{x}_0^*$, if the beacons' depths z_0^j are approximately equal, denote $z_0^*=\frac{1}{n_0}\sum_{j=1}^{n_0}z_0^j$, simpler formulas for $\tilde{\mathbf{x}}_0^{(1)},\Delta\tilde{\mathbf{x}}$ can be used:

$$\tilde{\mathbf{x}}_0^{(1)}=\mathbf{x}_0^*-\Delta\tilde{\mathbf{x}}_{\min}\,,\ \Delta\tilde{\mathbf{x}}=2\Delta\tilde{\mathbf{x}}_{\min}\,,$$
$$\Delta\tilde{\mathbf{x}}_{\min}=\sqrt{D_{\max}^2-(\max_{j=1,\ldots,n_0}|\mathbf{y}_0^j-\tilde{\mathbf{y}}_0|)^2-(z_0^*-z_0)^2}\,.$$

Step 2. Using the linearized representation of the measurement vector Y_0 and iterative algorithm to process them [22], estimates $\breve{\mathbf{x}}_0^{(l)},\breve{\mathbf{y}}_0^{(l)}$ are computed for each $l=1,\ldots,L$. Preliminary estimates $\tilde{\mathbf{x}}_0^{(l)},\tilde{\mathbf{y}}_0$ are used as a priori estimates and linearization points at the first iteration. Their errors are considered unknown parameters. The elements of the noise covariance matrix $\breve{R}_0$ of the measurement vector Y_0 are determined as

$$\breve{R}_0^{j,k}=T_0{}^kT_0{}^j\sigma_{\Delta c}^2+\sigma_{\delta_0}^2+\delta^{j,k}\sigma_v^2,\ j,k=1,\ldots,n_0\,. \quad (9)$$

The components of $T_0^j\Delta c$ refer to the measurement noise. Values of $T_0{}^j$ are computed depending on l. At the first iteration, expression $\sqrt{(\mathbf{x}_0^j-\tilde{\mathbf{x}}_0^{(l)})^2+(\mathbf{y}_0^j-\tilde{\mathbf{y}}_0^{(l)})^2+(z_0^j-z_0)^2}\big/\hat{c}_0$ is used for $T_0{}^j$, at further iterations the estimates $\breve{\mathbf{x}}_0^{(l)},\breve{\mathbf{y}}_0^{(l)}$ obtained at the previous iteration are used instead of $\tilde{\mathbf{x}}_0^{(l)},\tilde{\mathbf{y}}_0^{(l)}$. Further, $T_i{}^j$ is computed similarly using the estimates available at the i-th time.

The estimates $\breve{\mathbf{x}}_0^{(l)},\breve{\mathbf{y}}_0^{(l)},l=1,\ldots,L$ are concentrated near the points of the true and false AUV positions. The points with coordinates $\breve{\mathbf{x}}_0^{(l)},\breve{\mathbf{y}}_0^{(l)},l=1,\ldots,L$ are divided into two groups. They are assigned new indexing $\{l\}$ so that $\breve{\mathbf{x}}_0^{\{1\}}\leq\breve{\mathbf{x}}_0^{\{2\}}\leq\cdots\leq\breve{\mathbf{x}}_0^{\{L\}}$. The points having the coordinates $\breve{\mathbf{x}}_0^{\{l\}},\breve{\mathbf{y}}_0^{\{l\}}$ with the indexes $l=1,\ldots,L^{[1]}$, where $L^{[1]}=\arg\max_{l=1,\ldots,L-1}(\breve{\mathbf{x}}_0^{\{l+1\}}-\breve{\mathbf{x}}_0^{\{l\}})$, refer to the first group, and the points whose coordinate indexed with $l=L^{[1]}+1,\ldots,L$ refer to the second group. The coordinates of centers of these point groups $\begin{pmatrix}\breve{\mathbf{x}}_0^{[1]}\\ \breve{\mathbf{y}}_0^{[1]}\end{pmatrix}=\frac{1}{L^{[1]}}\sum_{l=1}^{L^{[1]}}\begin{pmatrix}\breve{\mathbf{x}}_0^{\{l\}}\\ \breve{\mathbf{y}}_0^{\{l\}}\end{pmatrix}$, $\begin{pmatrix}\breve{\mathbf{x}}_0^{[2]}\\ \breve{\mathbf{y}}_0^{[2]}\end{pmatrix}=\frac{1}{L-L^{[1]}}\sum_{l=L^{[1]}+1}^{L}\begin{pmatrix}\breve{\mathbf{x}}_0^{\{l\}}\\ \breve{\mathbf{y}}_0^{\{l\}}\end{pmatrix}$ are determined and recalculated to $\breve{x}_0^{[u]},\breve{y}_0^{[u]},u=1,2$ using the transformation inverse of (3).

Coordinate estimation procedure at the steps 1 and 2 is presented in Fig. 3, where the solid circles are the lines of position corresponding to range measurements, and the dashed circles are the boundaries of areas where signals from the relevant beacon may be received. The red line is the line of position for the measurement ρ^1 computed according to (4).

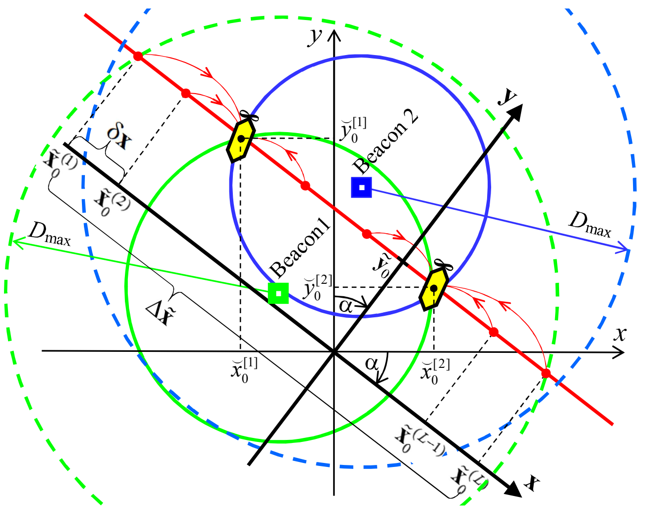

Fig. 3. Estimation of AUV coordinates at steps 1, 2 at the algorithm start in Case A.

Step 3. From the measurement vector Y_0, where ranges are linearized at the points $\breve{x}_0^{[u]}, \breve{y}_0^{[u]}, u=1,2$, the estimates of the four-dimensional vector

$$\chi = (x_0, y_0, \Delta c, b)^{\mathrm{T}}$$

and their error covariance matrices $\hat{\chi}^{[u]}, P_\chi^{[u]}, u=1,2$ are computed. Here $\breve{\chi}^{[u]} = \left(\breve{x}_0^{[u]}, \breve{y}_0^{[u]}, 0, 0\right)^{\mathrm{T}}$ serve as a priori estimates. The values of $T_0{}^j$ are also determined using $\breve{x}_0^{[u]}, \breve{y}_0^{[u]}$. The inverse error covariance matrix of estimates $\breve{\chi}^{[u]}$ for two u has the form $\breve{P}_\chi^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sigma_{\Delta c}^2 & 0 \\ 0 & 0 & 0 & 1/\sigma_b^2 \end{bmatrix}$, so the errors of estimates $\breve{x}_0^{[u]}, \breve{y}_0^{[u]}$ are considered unknown parameters as earlier. The elements of the noise covariance matrix R_0 of measurement vector Y_0 are

$$R_0^{j,k} = \sigma_e^2 + \delta^{j,k}\sigma_v^2,\ j,k = 1,\dots,n_0. \qquad (10)$$

Note that Δc, b are not estimated if $n_0 = 2$, because only two measurements and two unknown parameters are included in the problem. However, it makes sense to obtain the covariance matrices $P_\chi^{[u]}$ for the 4-dimensional vector χ, since they allow considering the mutual correlation between Δc, b, and estimation errors of x_0, y_0.

For further solution we form the estimates $\hat{X}_0^{[u]} = (\hat{\chi}^{[u]\mathrm{T}}, 0,0,0)^{\mathrm{T}}$ of the state vector and their block-diagonal error covariance matrices $P_0^{[u]}, u=1,2$ with the diagonal blocks $P_\chi^{[u]}$ and $\begin{bmatrix} \sigma_{\Delta K}^2 & 0 & 0 \\ 0 & \sigma_{\Delta U}^2 & 0 \\ 0 & 0 & \sigma_{\Delta U}^2 \end{bmatrix}$.

Case B: random desynchronization, $n_0 \geq 3$, $\operatorname{rank}\begin{pmatrix} x_0^1 & \cdots & x_0^{n_0} \\ y_0^1 & \cdots & y_0^{n_0} \\ 1 & \cdots & 1 \end{pmatrix} = 3$ – AUV receives signals from three and more beacons, whose position projections on the horizontal plane do not lie on a straight line. Here we consider the scenarios with favorable (B1 in Fig. 2) and unfavorable B2 in Fig. 2) beacon configurations for AUV positioning accuracy. In the second scenario, the beacons lie approximately on the same line.

To obtain the preliminary estimates of coordinates x_0, y_0, we compute

$$\begin{aligned} \rho^j &= [(Y_0^{j+1})^2 - (Y_0^1)^2 - (x_0^{j+1})^2 + (x_0^1)^2 - \\ &-(y_0^{j+1})^2 + (y_0^1)^2 - (z_0^{j+1} - z_0)^2 + (z_0^1 - z_0)^2]/2 \approx \\ &\approx \left[x_0^1 - x_0^{j+1} \;\middle|\; y_0^1 - y_0^{j+1}\right]\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \\ &+ Y_0^1(\Delta c T_0{}^1 + \delta_0 + v_0{}^1) - \\ &= Y_0^{j+1}(\Delta c T_0{}^{j+1} + \delta_0 + v_0{}^{j+1}),\ j = 1,\dots,n_0 - 1. \end{aligned} \qquad (11)$$

Here in the last expression (after ≈), as in (4), the part $[(\Delta c T_0{}^{j+1} + \delta_0 + v_0^{j+1})^2 - (\Delta c T_0{}^1 + \delta_0 + v_0^1)^2]/2$ is omitted. Using least squares method with (11) as the measurements, estimates of the coordinates are computed:

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix} = \tilde{P}_{x_0 y_0} \tilde{H}^{\mathrm{T}} \tilde{R}^{-1} \rho, \qquad (12)$$

where $\rho = \begin{pmatrix} \rho^1 \\ \vdots \\ \rho^{n_0 - 1} \end{pmatrix}$; $\tilde{H} = \begin{pmatrix} x_0^1 - x_0^2 & y_0^1 - y_0^2 \\ \vdots & \vdots \\ x_0^1 - x_0^{n_0} & y_0^1 - y_0^{n_0} \end{pmatrix}$; $\tilde{R}$ is the noise covariance matrix with the elements defined by (7); $\tilde{P}_{x_0 y_0} = (\tilde{H}^{\mathrm{T}} \tilde{R}^{-1} \tilde{H})^{-1}$ is the error covariance matrix of estimates $\tilde{x}_0, \tilde{y}_0$. Note that from the condition $\operatorname{rank}\begin{pmatrix} x_0^1 & \cdots & x_0^{n_0} \\ y_0^1 & \cdots & y_0^{n_0} \\ 1 & \cdots & 1 \end{pmatrix} = 3$ it follows that $\operatorname{rank}\tilde{H} = 2$, therefore, $\tilde{P}_{x_0 y_0}$ is not singular.

From matrix $\tilde{P}_{x_0 y_0}$ the major semiaxis a of the error ellipse is determined – square root from the larger matrix eigenvalue.

If $a < \bar{a}$, where $\bar{a}$ is the specified threshold, step 3 for Case A is performed with unambiguous AUV position, i.e., u = 1, and $\breve{x}_0, \breve{y}_0$ obtained in (12) are used instead of $\breve{x}_0^{[1]}, \breve{y}_0^{[1]}$. This situation occurs when the beacons do not lie close to one straight line (B1 in Fig. 2).

If $a \geq \bar{a}$, the observed beacons are located approximately on one straight line (B2 in Fig. 2), and here, like in Case A, we consider two hypotheses on AUV position. The angle α between the major axis of the error ellipse (eigenvector for the larger eigenvalue of matrix $\tilde{P}_{x_0 y_0}$) and axis x is determined. Then Step 1 is partially performed for the Case A: the beacons' coordinates **x**, **y** are determined using (3) and preliminary estimates $\tilde{x}_0^l, \tilde{y}_0^l, l = 1, \ldots, L$ are generated. Here the formula $\tilde{\mathbf{y}}_0 = \tilde{x}_0 \sin\alpha + \tilde{y}_0 \cos\alpha$ is used, and $\tilde{\mathbf{x}}_0^{(1)}, \Delta\tilde{\mathbf{x}}$ is computed using (8). Further, Steps 2, 3 for the Case A are performed.

Case C: unknown desynchronization, $n_0 \geq 3$, $\mathrm{rank}\begin{pmatrix} x_0^1 & \cdots & x_0^{n_0} \\ y_0^1 & \cdots & y_0^{n_0} \\ 1 & \cdots & 1 \end{pmatrix} = 2$ – measurements from three or more beacons are available, whose position projections on the horizontal plane lie on one straight line (C in Fig. 2). In this case, the algorithm is similar to the Case A with transformation from x, y to **x**, **y** by rotating the frame by angle α between axis y and the straight line passing through the beacon position horizontal projections. At Step 1 the estimate $\tilde{\mathbf{y}}_0$ is obtained using (6). Here, unlike Case A, we calculate not the noise covariance matrix $\tilde{R}$, but its inverse directly included in (6):

$$\tilde{R}^{-1} = \lim_{\sigma_{\delta_0} \to \infty} (\tilde{R}^* + \sigma_{\delta_0}^2 \Delta Y_0 \Delta Y_0^{\mathrm{T}})^{-1} = \\ = \lim_{\sigma_{\delta_0} \to \infty} \left(\tilde{R}^{*-1} - \frac{\tilde{R}^{*-1} \Delta Y_0 \Delta Y_0^{\mathrm{T}} \tilde{R}^{*-1}}{1/\sigma_{\delta_0}^2 + \Delta Y_0^{\mathrm{T}} \tilde{R}^{*-1} \Delta Y_0} \right) = \\ = \tilde{R}^{*-1} - \frac{\tilde{R}^{*-1} \Delta Y_0 \Delta Y_0^{\mathrm{T}} \tilde{R}^{*-1}}{\Delta Y_0^{\mathrm{T}} \tilde{R}^{*-1} \Delta Y_0}, \quad (13)$$

where $\Delta Y_0 = (Y_0^2 - Y_0^1, \ldots, Y_0^{n_0} - Y_0^1)^{\mathrm{T}}$; $\tilde{R}^*$ is the matrix with the elements determined as those of matrix $\tilde{R}$ in (7), but with $\sigma_{\delta_0} = 0$. Note that $\tilde{R}^{-1}$ is singular, i.e., $\tilde{R}$ does not have a finite value. Matrices $\breve{R}_0^{-1}$, R_0^{-1} used at the Steps 2, 3 are found similarly to $\tilde{R}^{-1}$:

$$\begin{aligned} \breve{R}_0^{-1} &= \breve{R}_0^{*-1} - \frac{\breve{R}_0^{*-1} J J^{\mathrm{T}} \breve{R}_0^{*-1}}{J^{\mathrm{T}} \breve{R}_0^{*-1} J} = \breve{R}_0^{*-1} - \frac{\breve{G} \breve{G}^{\mathrm{T}}}{\breve{g}}, \\ R_0^{-1} &= R_0^{*-1} - \frac{R_0^{*-1} J J^{\mathrm{T}} R_0^{*-1}}{J^{\mathrm{T}} R_0^{*-1} J} = R_0^{*-1} - \frac{G G^{\mathrm{T}}}{g}, \end{aligned} \quad (14)$$

where $\breve{R}_0^*$, R_0^* are the $(n_0–1) \times (n_0–1)$ matrices with the elements defined by (9) with $\sigma_{\delta_0} = 0$ and (10) with $\sigma_e = 0$; J is the $(n_0–1)$-dimensional vector of ones; $\breve{G}$, G are the $(n_0–1)$-dimensional vectors, whose elements are the sums of elements in the relevant rows of $\breve{R}_0^{*-1}$, R_0^{*-1}; $\breve{g}$, g are the sums of elements of $\breve{G}$, G.

At the Step 3 in this and two further cases D and E, χ is a three-dimensional vector (not four-dimensional, as in Case A): $\chi = (x_0, y_0, \Delta c)^{\mathrm{T}}$. The estimates of the state vector $\hat{X}_0^{[u]}$ and their block-diagonal error matrices $P_0^{[u]}, u = 1, 2$ are computed similarly to Case A.

Case D: unknown desynchronization, n_0 = 3, $\mathrm{rank}\begin{pmatrix} x_0^1 & x_0^2 & x_0^3 \\ y_0^1 & y_0^2 & y_0^3 \\ 1 & 1 & 1 \end{pmatrix} = 3$ – measurements from three beacons are available, whose position projections on the horizontal plane do not lie on a straight line. In these conditions, both unambiguous (D1 in Fig. 2) and ambiguous (D2 in Fig. 2) solutions are possible, following three steps from the Case A, but with some differences. The rotation angle α of the axes **x**, **y** relative to x, y in this case is such that

$$\sin\alpha \propto S = -\Delta Y_0^2 \Delta x_0^1 + \Delta Y_0^1 \Delta x_0^2, \\ \cos\alpha \propto C = -\Delta Y_0^2 \Delta y_0^1 + \Delta Y_0^1 \Delta y_0^2,$$

where $\Delta Y_0^j = Y_0^{j+1} - Y_0^1$, $\Delta x_0^j = x_0^{j+1} - x_0^1$, $\Delta y_0^j = y_0^{j+1} - y_0^1$, j = 1, 2. As a scalar measurement at Step 1, instead of ρ^1 in Case A, we use

$$\psi = \Delta Y_0^2 \rho^1 - \Delta Y_0^1 \rho^2 \approx \\ \approx -\begin{pmatrix} \Delta Y_0^2 & -\Delta Y_0^1 \end{pmatrix} \begin{pmatrix} \Delta x_0^1 & \Delta y_0^1 \\ \Delta x_0^2 & \Delta y_0^2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \cdots = \\ = \sqrt{S^2 + C^2}\, \mathbf{y_0} + \cdots,$$

where ρ^1, ρ^2 are determined by (11), '⋯' denote the components due to errors Δc, v_0^j, $j = 1,2,3$. The estimate $\tilde{\mathbf{y}}_0$ at Step 1 is calculated by the formula $\tilde{\mathbf{y}}_0 = \psi / \sqrt{S^2 + C^2}$ substituting (5), (6) in Case A. If at Step 2 the difference $\breve{\mathbf{x}}_0^{\{L\}} - \breve{\mathbf{x}}_0^{\{1\}}$ turns out to be less than the threshold, the values $\breve{\mathbf{x}}_0^{\{1\}},\ldots,\breve{\mathbf{x}}_0^{\{L\}}$ are not divided into two groups, $L^{[1]} = L$ is accepted, and only one hypothesis is considered. At Steps 2, 3, as in the previous case C, $\breve{R}_0^{-1}$, R_0^{-1} from (14) are used.

Case E: unknown desynchronization, $n_0 \geq 4$, $\operatorname{rank}\begin{pmatrix} x_0^1 & \cdots & x_0^{n_0} \\ y_0^1 & \cdots & y_0^{n_0} \\ 1 & \cdots & 1 \end{pmatrix} = 3$ – measurements from four or more beacons are available, whose position projections on the horizontal plane do not lie on a straight line. This is an analog of the case B. Here we consider the scenarios with favorable (E1 in Fig. 2) and unfavorable (E2 in Fig. 2) beacon configurations for AUV positioning accuracy. The unfavorable situation occurs when the beacons lie approximately on the same straight line. The actions as in Case B are performed, with the difference that $\tilde{R}^{-1}$ in (12) is determined using (13), and at Steps 2, 3 $\breve{R}_0^{-1}$, R_0^{-1} computed by (14) are used.

Note that the technique employed in cases C, D, E – processing of range measurements, each of which contains the same unknown desynchronization error between the signal receiver and emitter with a special inverse measurement noise covariance matrix – has earlier been applied in [13]. Alternatively, at Steps 2, 3 in Cases C, D, E the vector of difference measurements ΔY_0 with the relevant observation matrix and noise covariance matrix can be processed.

After obtaining the coordinate estimates $\breve{x}_0^{[u]}, \breve{y}_0^{[u]}$ for AUV position hypotheses $u = 1, 2$ at Step 2, they can be checked for compliance with the maximal beacon signal reception range $D_{\max}$. If the range from the point $\breve{x}_0^{[u]}, \breve{y}_0^{[u]}$ to at least one used beacon exceeds $D_{\max}$, the hypothesis u can be rejected, and the other hypothesis u^* can be considered true and its position estimates may be taken to be the unambiguous solution.

It should be remembered that in case of ambiguous solution under small range between AUV possible solutions compared with the measurement errors, the distribution of position estimate errors greatly differs from the Gaussian one. This does not allow the effective use of Kalman type algorithms based on Gaussian approximation of the state vector a posteriori density. These situations are left beyond the scope of this paper.

3. FUSION OF SAVED AND CURRENT MEASUREMENTS IN ONE FILTER

Therefore, after obtaining $\hat{X}_0^{[u]}$, $P_0^{[u]}$ for one ($u = 1$) or two ($u = 1, 2$) AUV position hypotheses, AUV coordinates x_i, y_i at the following times $i = 1, 2\ldots$ should be estimated with account of the newly coming measurements $Y_1, \ldots, Y_i$ and those saved before the algorithm start $Y_{-N^k}, \ldots, Y_{-1}$, $k = 1, 2$ is the type of desynchronization δ_i. This is done using one or two – depending on the number of hypotheses u – extended or iterated KF, which estimate the augmented state vector

$$\mathbb{X}_{s,i} = \left(x_s, y_s, \Delta K_s, \Delta U_s^x, \Delta U_s^y, X_i^{\mathrm{T}}\right)^{\mathrm{T}},$$

where s is the time before the algorithm start, successively decreasing from -1 to $-N^k$, i is the current time. The time i increases when s reaches $-\Delta N_0$ with $i = 0$ or the sum $s + \Delta N_0$ becomes divisible by ΔN with $i \geq 1$, where the values $\Delta N_0 \geq 0$ and $\Delta N \geq \max(1, \Delta N_0)$ are set based on the onboard computer performance and the fact that processing of Y_0 is more complicated as compared to other Y_i. Note that in the vector $\mathbb{X}_{s,i}$ the values x, y, ΔK, ΔU^x, ΔU^y are presented for two moments i and s, while constants Δc, b are presented only once.

The diagram of processing the current measurements (with indices $i = 1, 2\ldots$) and saved measurements (with indices $s = -1, \ldots, -N^k$) is shown in Fig. 4.

With random δ_i ($k = 1$), range measurement vectors Y_i, Y_s participate in the processing, and with unknown δ_i ($k = 2$), vector $\Delta Y_i = (\Delta Y_i^1, \ldots, \Delta Y_i^{n_i-1})^{\mathrm{T}}$ of difference measurements $\Delta Y_i^j = Y_i^{j+1} - Y_i^1$ and the similarly formed ΔY_s.

It should be explained that with $k = 2$, $i > 0$ the KF may use the original range measurements Y_i, Y_s and inverse noise covariance matrix of a special form as described in the previous section for $i = 0$. But the thing is that with $i = 0$, the three-dimensional vector $\chi = (x_0, y_0, \Delta c)^{\mathrm{T}}$ is to be estimated, whereas with $i > 0$ we estimate the state vector with a dimensionality greatly exceeding that of the measurement vector, and this technique turns more labor-consuming compared to processing of difference measurements ΔY_i, ΔY_s.

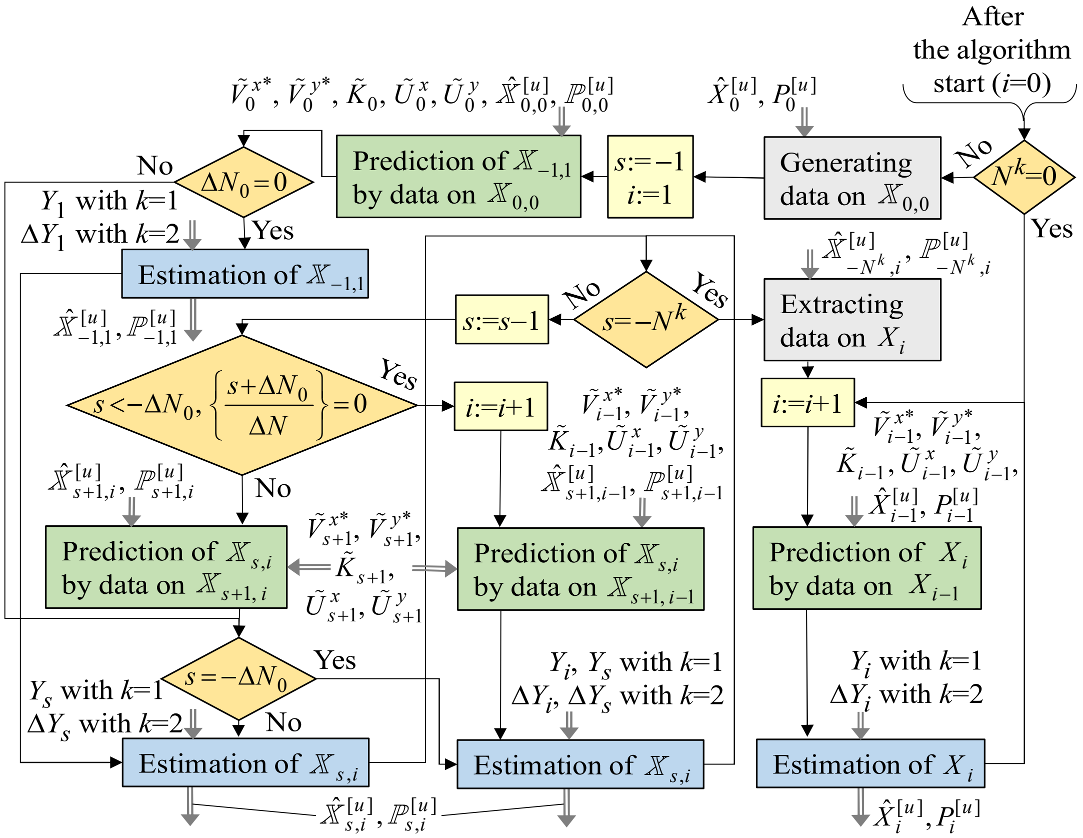

Fig. 4. The diagram of processing the current and saved measurements; $\{\cdot\}$ is the fractional part.

Coordinates in forward time (x_i, y_i, $i \ge 1$) are predicted using (2), and in backward time (x_s, y_s, $s \le -1$), using

$$\begin{aligned} x_s &= x_{s+1} - (\tilde V^y_{s+1}\Delta K_{s+1} + \tilde V^x_{s+1} + \tilde U^x_{s+1} + \Delta U^x_{s+1})\Delta t_{s+1} - w^x_{s+1}, \\ y_s &= y_{s+1} - (-\tilde V^x_{s+1}\Delta K_{s+1} + \tilde V^y_{s+1} + \tilde U^y_{s+1} + \Delta U^y_{s+1})\Delta t_{s+1} - w^y_{s+1}. \end{aligned} \quad (15)$$

Considering the stationarity of ΔK, ΔU^x, ΔU^y, prediction of $\Delta K_s, \Delta U^x_s, \Delta U^y_s$ in backward time is made with the same equations as $\Delta K_i, \Delta U^x_i, \Delta U^y_i$ in forward time, with the difference that variables in the right part of equations for $\Delta K_s, \Delta U^x_s, \Delta U^y_s$ have the index $s + 1$.

All components of the augmented state vector $\mathbb{X}_{s,i}$ are predicted with ($i = 1$, $s = -1$) or $\left(s<-\Delta N_0, \left\{\frac{s+\Delta N_0}{\Delta N}\right\}=0\right)$. If the latter condition is not met, only five first components of $\mathbb{X}_{s,i}$ referring to the time s are predicted.

When s reaches $-N^k$, i.e., after processing of all the saved measurements, the augmented state vector $\mathbb{X}_{-N^k,i}$ is substituted with the original state vector X_i. Then the estimates $\hat X^{[u]}_i$ and their error covariance matrices $P^{[u]}_i$ are extracted from $\hat{\mathbb{X}}^{[u]}_{-N^k,i}$, $\mathbb{P}^{[u]}_{-N^k,i}$. Further, KF predicts and estimates X_i by the current measurements Y_i or ΔY_i. If $N^k = 0$, such KF is activated immediately after obtaining $\hat X^{[u]}_0$, $P^{[u]}_0$.

To understand what measurement set has been processed with KF by the current time, Fig. 5 presents the transformations of Gaussian approximations of a posteriori densities $\mathbf{f}\left(\begin{matrix}\text{state}\\ \text{vector}\end{matrix}\middle|\begin{matrix}\text{processed}\\ \text{measurements}\end{matrix}\right)$ with $N^k > 0$, i.e., when saved measurements are available. This diagram, similarly to the one in Fig. 4, reflects the KF performance for one of hypotheses u. It shows the range measurements. With $k = 2$ these are the measurements from which the range difference measurements are formed.

As $\mathbb{X}_{s,i}$ is estimated, the list of processed measurements to the right of the line in $\mathbf{f}(\cdot|\cdot)$ in the grey part of the diagram increases up and down, whereas as X_i is estimated, it increases only down. Each vertical column of transformations $\mathbf{f}(\cdot|\cdot)$ in the shaded area denotes the cycle of processing of a certain number of saved measurements (Y with negative indices) and one current measurement (Y with positive index). At the same time, at the first cycle the saved measurements are not processed if $\Delta N_0 = 0$, and at the last cycle the current measurement is not processed if $(N^k - \Delta N_0) / \Delta N$ is not an integer number.

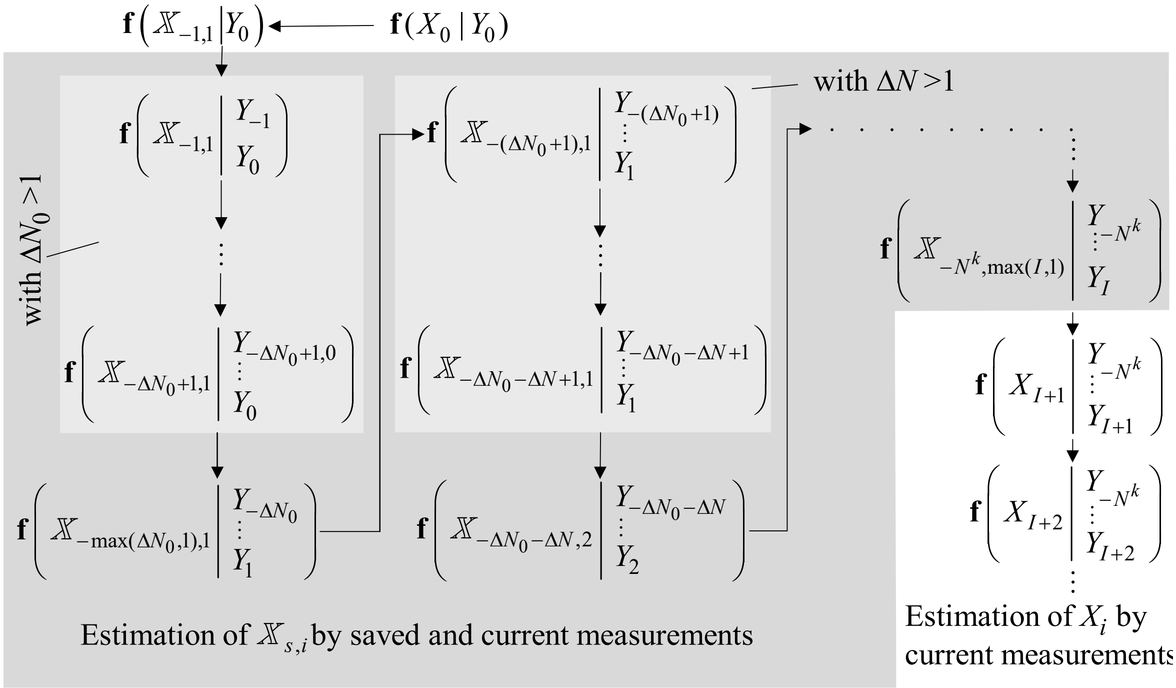

Fig. 5. Diagram of transformations of Gaussian approximations of a posteriori densities during processing of saved and current measurements, where $I = \lfloor (N^k - \Delta N_0)/\Delta N + 1 \rfloor$, $N^k > 0$, $\lfloor . \rfloor$ is rounding downward.

Below we provide a brief comment on (2), (15) for predicting the coordinates in forward and backward time linearized relative to the heading indicator readings $\tilde{K}$. Under highly accurate heading generation this is acceptable, but with large heading errors ΔK it is rational to linearize the equations relative to corrected heading $\hat{K}$ with account for the estimates obtained before prediction. Then ΔK in the equations should be understood as an error of corrected heading $\hat{K}$ rather than original heading $\tilde{K}$. The equation for ΔK does not change.

When solving the problem with two hypotheses $u = 1, 2$ for $i \geq 1$, as at the algorithm start ($i = 0$), we can check if the range from the points $\hat{x}_i^{[u]}, \hat{y}_i^{[u]}$ to some of the used beacons exceeds $D_{\max}$. If the threshold is exceeded for one hypothesis, it is considered false, its KF stops working, and the other hypothesis is taken as a true one u^*. If the threshold is not exceeded, the true hypothesis is selected based on stochastic approach as shown in the next section.

4. AMBIGUITY RESOLUTION

When solving the problem with two hypotheses on AUV position $u = 1, 2$ consider their a posteriori probabilities $\mathbf{p}(u|\mathbf{Y}_i)$, i.e., conditional probabilities of u given the vector $\mathbf{Y}_i$ consisting of all measurements $\mathbf{Y}_m$ processed by the time i, where $m = 0$ with $i = 0$, $m = -\min(\Delta N_0 + (i-1)\Delta N, N^k)$, …, i with $i \geq 1$. With unknown δ_i ($k = 2$) by $\mathbf{Y}_m$ we mean the range measurements participating in the formation of range difference measurements. Remind that if with $k = 2$ only one range measurement is available for some time, it is not used and is not included in Y_m. The ratio of a posteriori probabilities of the first and second hypotheses $\delta p_i^{1/2} = \dfrac{\mathbf{p}(u=1\,|\,\mathbf{Y}_i)}{\mathbf{p}(u=2\,|\,\mathbf{Y}_i)}$ allows selecting the true hypothesis u^* according to the rule

$$u^* = \begin{cases} 1, \text{ with } \delta p_i^{1/2} \geq \delta\overline{p}, \\ 2, \text{ with } \delta p_i^{1/2} \leq 1/\delta\overline{p}, \\ \text{not determined, with } 1/\delta\overline{p} < \delta p_i^{1/2} < \delta\overline{p}, \end{cases}$$

where $\delta\overline{p} \gg 1$ is the specified threshold. Therefore, out of $u = 1, 2$ hypotheses, the one with a posteriori probability larger than that of alternative hypothesis by a specified number of times is considered to be true.

The ratio of a posteriori probabilities $\delta p_i^{1/2}$ is determined with an expression based on the Gaussian distribution of random values included in the problem with equal a priori probabilities of hypotheses $u = 1, 2$:

$$\delta p_i^{1/2} = \sqrt{B_i}\, e^{A_i/2},$$

where

$$A_0=\begin{cases}0, \text{ with } n_0 = k+1,\\ \vartheta_0^{[2]\mathrm{T}}(\Theta_0^{[2]})^{-1}\vartheta_0^{[2]}-\vartheta_0^{[1]\mathrm{T}}(\Theta_0^{[1]})^{-1}\vartheta_0^{[1]}, \text{with } n_0>k+1,\end{cases}$$

$$B_0=\frac{\left|P_\chi^{[1]}\right|}{\left|P_\chi^{[2]}\right|}, (\Theta_0^{[u]})^{-1}=R_0^{-1}-R_0^{-1}H_0^{[u]}P_\chi^{[u]}H_0^{[u]\mathrm{T}}R_0^{-1},$$

$P_\chi^{[u]}$ is the error covariance matrix of estimating the 4-dimensional vector χ by Y_0 (with $k = 1$) or the 3-dimensional vector χ by ΔY_0 (with $k = 2$), $\vartheta_0^{[u]}$, $H_0^{[u]}$, and R_0 are the residual vector, observation matrix and measurement noise covariance matrix used in vector χ estimation by Y_0 or ΔY_0;

$$A_i = A_{i-1}+\delta A_i^{[2]}-\delta A_i^{[1]}+\Delta A_i^{[2]}-\Delta A_i^{[1]};$$
$$B_i = B_{i-1}\delta B_i^{[2]}\Delta B_i^{[2]}\big/\left(\delta B_i^{[1]}\Delta B_i^{[1]}\right),\ i\ge 1;$$

$$\delta A_i^{[u]}=\begin{cases}\sum_s \vartheta_{s,i}^{[u]\mathrm{T}}(\Theta_{s,i}^{[u]})^{-1}\vartheta_{s,i}^{[u]},\\ \quad\text{with } (i>1 \text{ or } \Delta N_0>0) \text{ and } i\le I,\\ 0, \text{ with } (i=1 \text{ and } \Delta N_0=0) \text{ or } i>I,\end{cases}$$

$$\delta B_i^{[u]}=\begin{cases}\prod_s \left|\Theta_{s,i}^{[u]}\right|,\\ \quad\text{with } (i>1 \text{ or } \Delta N_0>0) \text{ and } i\le I,\\ 1, \text{ with } (i=1 \text{ and } \Delta N_0=0) \text{ or } i>I,\end{cases}$$

$$s=-\max(\Delta N_0+(i-2)\Delta N, 0)-1,\ldots,-$$
$$-\min(\Delta N_0+(i-1)\Delta N, N^k);$$

$$\Delta A_i^{[u]}=\begin{cases}0, \text{ with } \left((i>1 \text{ or } \Delta N_0>0) \text{ and } i<I\right)\\ \text{or } i=I=\tilde I,\\ \vartheta_i^{[u]\mathrm{T}}(\Theta_i^{[u]})^{-1}\vartheta_i^{[u]}, \text{ with } i>I\\ \text{or } i=I<\tilde I \text{ or } (i=1 \text{ and } \Delta N_0=0),\end{cases}$$

$$\Delta B_i^{[u]}=\begin{cases}1, \text{ with } \left((i>1 \text{ or } \Delta N_0>0) \text{ and } i<I\right)\\ \text{or } i=I=\tilde I,\\ \left|\Theta_i^{[u]}\right|, \text{ with } i>I\\ \text{or } i=I<\tilde I \text{ or } (i=1 \text{ and } \Delta N_0=0),\end{cases}$$

$\tilde I=\dfrac{N^k-\Delta N_0}{\Delta N}+1$, $I=\begin{cases}0, \text{ with } N^k=0;\\ \lfloor\tilde I\rfloor, \text{ with } N^k>0\end{cases}$ is the value of i when the processing of saved measurements stops, $\lfloor\cdot\rfloor$ is rounding downward; $\vartheta_i^{[u]}$ and $\Theta_i^{[u]}$ for $i > 0$ is the residual vector of measurements Y_i or ΔY_i by the prediction of X_i based on $\mathbf{Y}_{i-1}$ and its covariance matrix. Interpretation of $\vartheta_{s,i}^{[u]}$, $\Theta_{s,i}^{[u]}$ depends on s, i. If $i = 1$, $-\Delta N_0 < s$ or $i > 1$, $\{(s + \Delta N_0) / \Delta N\} \neq 0$, where $\{\cdot\}$ is the fractional part, $\vartheta_{s,i}^{[u]}$ and $\Theta_{s,i}^{[u]}$ denote the residual of scalar measurement Y_s or ΔY_s based on the prediction of $\mathbb{X}_{s,i}$ and its variance, otherwise, it is a residual vector of measurements Y_s, Y_i or ΔY_s, ΔY_i based on the prediction of $\mathbb{X}_{s,i}$ and its covariance matrix. In the latter case we mean the prediction of $\mathbb{X}_{s,i}$ by the measurements $Y_{s+1}, \ldots, Y_{i-1}$ or $\Delta Y_{s+1}, \ldots, \Delta Y_{i-1}$. Note that the measurement residuals and their covariance matrices included in the formulas above are used in KF and need not be specially computed. The formulas given in this section are based on the multiple model filtering theory [21, 31]. Calculating the ratio of a posteriori probabilities for two hypotheses with a described technique excludes the errors due to the bit grid limitation, which typically occur in direct calculation of a posteriori probabilities.

Once u^* is determined, KF for this hypothesis continues running, and KF for the alternative hypothesis is stopped.

CONCLUSIONS

The paper presents a computationally simple recursive AUV positioning algorithm based on measurements of ranges to acoustic beacons, electromagnetic speed log and heading indicator data. Getting simultaneous measurements from two beacons (with random desynchronization between the beacon and AUV clocks) or from three beacons (with unknown desynchronization) is sufficient for the algorithm to start. AUV a priori coordinates are not needed.

The algorithm starting procedure depends on the number and arrangement of beacons, and desynchronization clock type. Initial (comparatively coarse) estimation of AUV two or one coordinate in specified direction is common for all cases. This is done using the differences of squared range measurements to different beacons, with excluded squared AUV coordinates and squared desynchronization. Some simplifying assumptions are made, however, SDs of measurement noises and sound speed error are considered. The resultant initial solution that can be unambiguous or ambiguous (with two possible AUV positions) is found by processing the range or range difference measurements presented in linearized form with account of earlier coarse coordinate estimates.

Further, the current measurements and measurements saved before the algorithm start are processed together in an extended KF, with the saved measurement processed in reverse order. The amount of the saved measurements processed between the arrivals of new measurements depends on the computer performance. The state vector estimated by the KF includes the current coordinates, constant error of the sound speed, heading indicator error and errors in knowledge of current speed components as stationary processes, and with random desynchronization, constant bias of beacon and AUV clocks. When the saved measurements are processed, the state vector is augmented with the coordinates, heading indicator error, and errors in knowledge of current speed components at the time when these measurements were obtained.

The algorithm provides ambiguity resolution between two hypotheses on the AUV position, and applies the extended KF with a specific measurement linearization point to each hypothesis. Out of two hypotheses, the one with a priori probability larger than that of alternative hypothesis by a specified number of times is considered to be true. The ratio of a posteriori probabilities is computed by the outputs of two KFs.

The presented algorithm processing the saved (before the algorithm start) measurements in backward time can be modified for the other models of errors of autonomous sensors and acoustic measurements, including the sound speed error. As noted in the Introduction, the modified algorithm can be applied to other navigation applications using dead-reckoning and measured ranges or range differences to beacons or point landmarks.

The second part of the paper will present the results of studying the algorithm performance in terms of runtime and accuracy.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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