ACCURACY OF SENSOR BIAS ESTIMATION AND ITS RELATIONSHIP WITH ALLAN VARIANCE

O.A. Stepanov[1]
Concern CSRI Elektropribor, JSC, ITMO University, St. Petersburg, Russia

I.B. Chelpanov[2]
State Polytechnical University, St. Petersburg, Russia

A.V. Motorin[3]
Concern CSRI Elektropribor, JSC, ITMO University, St. Petersburg, Russia

Abstract

Key words: sensor errors; parameter identification, Allan variance

The paper discusses the relationship between Allan variance and error variance of sensor bias estimation obtained by averaging over a certain time interval. Allan variance is shown to coincide with this variance in some cases. Thus, Allan variance plots can be used to predict the accuracy of bias estimation, which is critical for the sensors whose signals are integrated in inertial systems. Improving of bias estimation accuracy using nonlinear filtering methods is discussed.

Introduction

Identification of sensor error model and determination of its parameters form an important problem to be solved by tests and calibration. Traditionally, algorithms for determining the spectral densities and correlation functions are used to design the model of error random components [1-6]. Allan variance method is also extensively used [7-13]. New methods are searched for, based, for example, on nonlinear filtering methods [14-17]. Determination of time-invariant error components (random bias) is also important, especially when the signals of sensors incorporated in IMUs are integrated and thus lead to accumulation of errors. Bias is often determined by usual averaging of sensor errors over a finite time period. Then the question arises, how the averaging time should be rationally selected so that for example error variance of the obtained estimate be minimum. On the other hand, the estimate obtained by averaging obviously will not be optimal (in terms of minimum error variance) if nonwhite noise components of sensor errors are present. Therefore, bias estimation accuracy can also be improved by using more advanced algorithms which are not reduced to simple averaging but account for the additional error components. As is known, Allan variance is insensitive to the presence of bias, since error increments rather than errors are used to calculate the Allan variance. However, Allan variance plots are still used to estimate the so called bias instability [18, 19], which is actually associated with the problem of estimating the random bias. Discussion of these issues is given in the paper.

Estimation accuracy of the sensor bias by averaging. Connection with Allan variance

Let the sensor error $z(t)$ be measured, which can be described in the form

$$y(t) = c + z(t); \tag{1}$$

where $z(t)$ is a nonstationary zero-mean random process, c is the random bias. It is required to estimate c using measurements $y(t)$. This kind of problem often occurs in sensor calibration performed at the test bench or in comparison of their signals with a reference more precise sensor. It is often solved by simple averaging of the measured error over finite time interval τ, i.e.

$$\hat{c}_\tau = \frac{1}{\tau}\int_0^\tau y(t)dt = c + \frac{1}{\tau}\int_0^\tau z(t)dt . \tag{2}$$

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[1] D. Sci., Professor. Head of Scientific and Educational Center, CSRI Elektropribor.
[2] D. Sci., Professor.
[3] Postgraduate.

Obviously, the following can be written for the estimate error and its variance:

$$\hat{c}_\tau - c = \frac{1}{\tau}\int_0^\tau z(t)dt,$$
$$M\left\{(\hat{c}_\tau - c)^2\right\} = M\left\{\left(\frac{1}{\tau}\int_0^\tau z(t)dt\right)^2\right\}. \tag{3}$$

Considering variance (3) to be the variance of increments of process $\frac{1}{\tau}\int_0^\tau z(t)dt$ and denoting $\overline{z}(t,\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} z(t^*)dt^*$, we get

$$M\left\{(\hat{c}_\tau - c)^2\right\} = M\left\{(\overline{z}(t+\tau,\tau) - \overline{z}(t,\tau))^2\right\}. \tag{4}$$

Assume that the following limiting relationship holds true:

$$M\left\{(\hat{c}_\tau - c)^2\right\} = \frac{1}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}(\overline{z}(t+\tau,\tau) - \overline{z}(t,\tau))^2\,dt, \tag{5}$$

meaning that calculation of mathematical expectation in (4) can be replaced with time averaging for one sample.

It can be easily seen that (5) coincides with the Allan variance [6, 8, 23]. Therefore, Allan variance coincides with the error variance of bias estimation calculated by averaging if (5) is valid for process $\overline{z}(t,\tau)$. Then optimal averaging time can be determined by Allan variance minimum point, and thus, corresponding minimum error variance of bias estimate found by averaging. Respectively square root of the Allan variance also called the Allan deviation [8,9,11] is the same as the root mean square (RMS) for the bias estimation error. The established relation seems helpful because it lets assess the accuracy of bias estimation by Allan variance plots insensitive to bias.

Consider an example. Let

$$y(t) = x(t) + \rho v(t), \tag{6}$$

where $x(t)$ is the random walk (Wiener process) set in the form $\dot{x} = qw$, $x(0) = 0$; qw, $\rho v(t)$ are independent zero-mean white noises with power spectral densities (PSD) q^2 and ρ^2, noises $w(t)$, $v(t)$ have unit PSDs. In other words, process $y(t)$ is a sum of random walk and white noise. Searching for estimate in the form (2), we can write:

$$\hat{c}_\tau - c = \frac{1}{\tau}\int_0^\tau x(t)dt + \frac{\rho}{\tau}\int_0^\tau v(t)dt. \tag{7}$$

It can be easily seen that the following relation is true for estimate error variance:

$$M\left\{(c - \hat{c}_\tau)^2\right\} = \frac{q^2}{\tau^2}M\left\{\left(\int_0^\tau\int_0^\tau w(t)dtdt\right)^2\right\} + \frac{\rho^2}{\tau^2}M\left\{\left(\int_0^\tau v(t)dt\right)^2\right\}. \tag{8}$$

Note that squared first and second integrals of white noise are under the signs of mathematical expectation in (8). Therefore, these mathematical expectations determine the variances of the first and second integrals. Using the known expressions for these variances [6], we obtain the expression

$$\sigma_{\Delta c}^2 = M\left\{(c - \hat{c}_\tau)^2\right\} = \frac{q^2\tau}{3} + \frac{\rho^2}{\tau}, \tag{9}$$

coinciding with the Allan variance for the sum of random walk and white noise. Differentiating (9) with respect to τ and setting the derivative equal to zero provides optimal (in terms of minimum variance) average time and corresponding minimum estimate variance:

$$\tau_{opt} = \frac{\sqrt{3}\rho}{q}, \quad \sigma_{\Delta c(\min)}^2 = \frac{2\rho q}{\sqrt{3}}. \tag{10}$$

Therefore averaging time optimal in the given sense is directly proportional to the square root of the ratio between white noise PSD and PSD of generating noise of random walk, and error variance is directly proportional to their product.

This example is illustrated by the simulation. Figure 1 shows real bias estimation RMS error calculated using 500 samples for six various time intervals vs Allan deviation for one sample. Note that the calculated from finite length sample AV is the estimate the true AV (5). This explains the incomplete match graphs in Figure 1, especially for large averaging time.

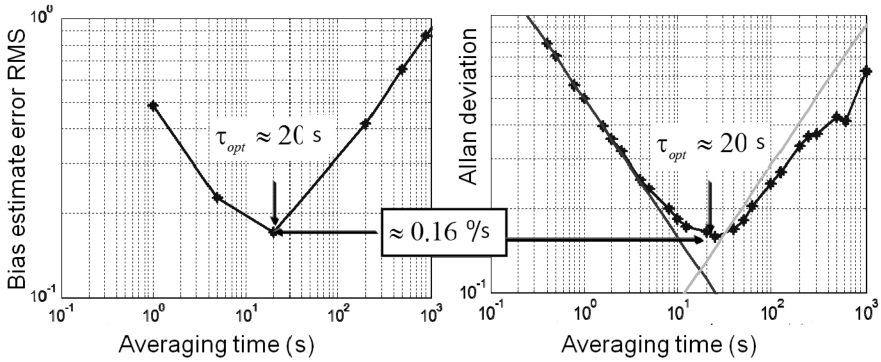

Fig. 1. Real RMS: bias estimation by averaging (left) and Allan deviation (right).

Note that the location of Allan variance minimum point in this statement depends on the ratio between PSDs of white noise and generating noise of random walk, as follows from (10). This is also illustrated in Fig. 2 showing Allan variances for four variants of error components

$$y_{11}(t) = x_1(t) + \rho_1 v(t); y_{12}(t) = x_1(t) + \rho_2 v(t); y_{21}(t) = x_2(t) + \rho_1 v(t); y_{22}(t) = x_2(t) + \rho_2 v(t),$$

where $\dot{x}_1(t) = q_1 w(t); \quad \dot{x}_2(t) = q_2 w(t)$.

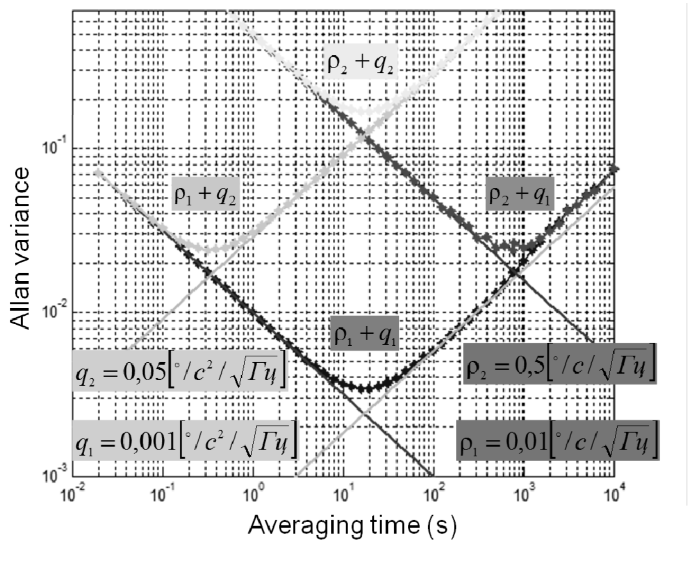

Fig. 2. Allan variance for sum of white noise and random walk of PSDs ρ_1^2, ρ_2^2, and q_1^2, q_2^2 respectively

Standards [18, 19] introduce the stability as “a measure of the ability of a sensor performance coefficient to remain invariant when continuously exposed to a fixed operating condition”. Note that no quantitative measures determining this ability are given. Let us discuss the possibility of introducing such quantitative measure for random bias of sensor error model (1). Obviously, in this case, the bias non stability is determined by the PSD of random walk generating noise q. In [9, 10, 18, 19] the minimum of AV plot is used to characterize bias instability in assumption that it is the flicker noise PSD. As follows of the aforementioned results, the AV may have an extremum in the absence of flicker noise in the error model. Thus the quantitative measure of stability depends on the model. For the model (1) the maximum averaging time leading to an increase in the bias estimation accuracy, and the corresponding RMS can be considered as the bias stability characteristic as well. However, these values also depend on the white noise PSD (as shown in Fig. 2) and does not consider the possibility of bias estimation by other methods, which are discussed below.

Improving the accuracy of bias estimation using nonlinear filtering

As mentioned in the Introduction, [15, 16] propose an approach based on nonlinear filtering for identification of sensor error models. Its idea lies in finding an optimal Bayesian estimate of composite filter including the state subvector of shaping filter of the studied process and the subvector of unknown parameters specifying this shaping filter. Following these references, formulate the statement of bias estimation problem with inaccurately known parameters of measurement error models. Introduce a composite vector $\tilde{x}_i^T = \left[X_i, \theta\right]^T$, where $X_i = [x_i, c]^T$, $\theta = [q, \rho]^T$, then nonlinear filtering problem in discrete form can be written as

$$\begin{aligned} x_i &= x_{i-1} + q\sqrt{\Delta t}w_i,, \\ c_i &= c_{i-1}=c, \\ q_i &= q_{i-1} = q, \\ \rho_i &= \rho_{i-1} = \rho, \\ y_i &= c_i + x_i + (\rho/\sqrt{\Delta t})v_i, \end{aligned} \tag{11}$$

where w_i and v_i are zero-mean Gaussian white noise sequences with unit variance, Δt is the sampling interval.

Introducing probability distribution function (PDF) $f(\theta)$ for vector θ and applying partitioning method (Rao-Blackwellization method), the following can be written for optimal estimate $\hat{\theta}_i(Y_i)$ and corresponding computational covariance matrix $P_i^{\theta}(Y_i)$ [14, 20]:

$$\hat{\theta}_i(Y_i) = \int \theta f(\theta/Y_i)d\theta,\ P_i^{\theta}(Y_i) = \int(\theta-\hat{\theta}_i)(\theta-\hat{\theta}_i)^{\mathrm{T}} f(\theta/Y_i)d\theta, \tag{12}$$

where $Y_i = \left[y_1,\dots,y_i\right]$ is the vector of measurements obtained by the time i. A posteriori PDF $f(\theta/Y_i)$ is defined as

$$f(\theta/Y_i) = \frac{f(\theta)f(Y_i/\theta)}{\int f(\theta)f(Y_i/\theta)d\theta}, \tag{13}$$

where $f(Y_i/\theta) = f(y_i/Y_{i-1},\theta)f(y_{i-1}/Y_{i-2},\theta)\dots f(y_1/\theta)$ is the likelihood function.

The distinctive feature of the problem is that with fixed $\theta = \theta^j$, Eqs. (11) set the linear gaussian filtering problem, and therefore PDFs

$$f(y_i/Y_{i-1},\theta=\theta^j) = N\left((y_i; H\hat{X}_{i/i-1}(\theta^j), D_i^{cond}(\theta^j)\right), \tag{14}$$

where $H = \begin{bmatrix}1 & 1\end{bmatrix}$; are also Gaussian. Incoming optimal prediction estimates $\hat{X}_{i/i-1}(\theta^j)$ and measurement residual variances $D_i^{cond}(\theta^j) = M\left\{\left(y_i - H\hat{X}_{i/i-1}(\theta^j)\right)^2\right\}$ are calculated using the bank of Kalman filters. To calculate the optimal estimate and conditional covariance matrix (12), the point-mass method can be used. Then it is implied that a priori PDF $f(\theta)$ is approximated as [14, 22]

$$f(\theta) = \sum_{j=1}^{L}\mu_0^j\delta(\theta-\theta^j),\ \mu_0^j = \frac{f(\theta=\theta^j)}{\sum_{j=1}^{L} f(\theta=\theta^j)}, \tag{15}$$

where θ^j, $j=\overline{1.L}$ is the set of possible values of parameters defining the point masses. Substituting (15) to (13), the following expressions can be written for a postreriori PDF $f(\theta/Y_i)$:

$$f(\theta/Y_i) = \sum_{j=1}^{L}\mu_i^j\delta(\theta-\theta^j), \qquad \mu_i^j = \frac{\mu_{i-1}^j\cdot f(y_i/Y_{i-1},\theta=\theta^j)}{\sum_{j=1}^{L}\mu_{i-1}^j f(y_i/Y_{i-1},\theta=\theta^j)}. \tag{16}$$

With account for (12), the following relationships can be easily obtained for the estimates and conditional covariance matrix:

$$\hat{\theta}_i(Y_i) \approx \sum_{j=1}^{L}\mu_i^j\theta_i^j, \qquad P_i^{\theta}(Y_i) \approx \sum_{j=1}^{L}\mu_i^j\theta_i^j(\theta_i^j)^T - \hat{\theta}_i\hat{\theta}_i^{\mathrm{T}}. \tag{17}$$

Nonlinear filtering can be used to get the bias estimate and its variance in the form

$$\hat{c}_i(Y_i) \approx \sum_{j=1}^{L} \hat{c}_i^j \mu_i^j , \qquad P_i^c(Y_i) \approx \sum_{j=1}^{L} P_i^{cj} \mu_i^j , \tag{18}$$

where $\hat{c}_i^j$; P_i^{cj} are the bias estimates and variances of obtained in each local Kalman filter.

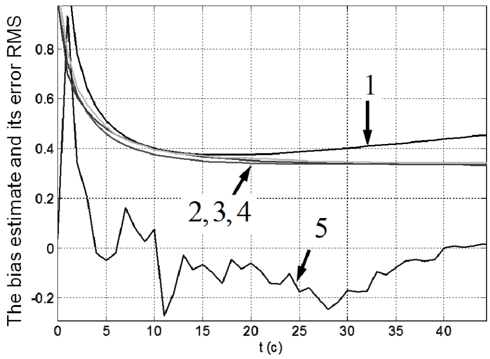

Fig. 3. Allan deviation (*1*), bias estimation error RMS for optimal KF (*2*), calculated (*3*) and real (*4*) bias estimation error RMS and bias estimation error sample (*5*) for adaptive filter

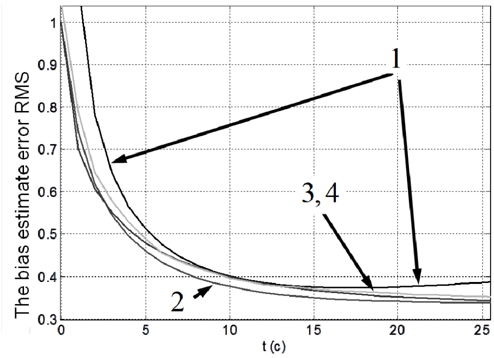

Fig. 4. Allan deviation (*1*), bias estimation error RMS for optimal KF (*2*), calculated (*3*) and real (*4*) bias estimation error RMS for adaptive filter

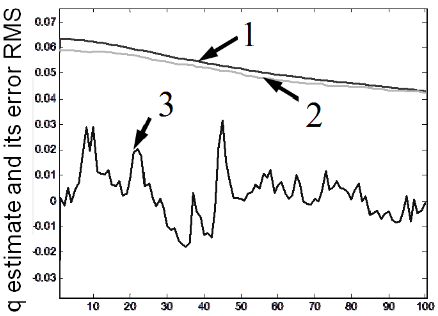

Fig. 5. Calculated (*1*) and real *(2)* estimation error RMS for q ; estimation error sample (*3*)

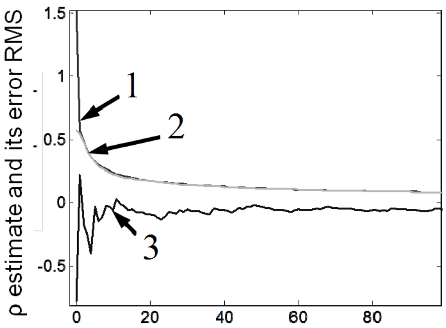

Рис. 6. Calculated (*1*) and real (*2*) estimation error RMS for ρ ; estimation error sample (*3*)

The efficiency of the adaptive filtering method have been proved by simulation. The values q and ρ that are determined noise PSDs were assumed as $q_{ad} \in [0.01 \quad 0.21]$ $\rho_{ad} \in [0.1 \quad 2.1]$, and initial bias RMS was 1. The simulation results are shown on figures 3-6. In comparison the Allan deviation plot and the RMS error of optimal Kalman Filter (KF) bias estimates is shown on figures 3, 4 for $q_{opt} = M\{q_{ad}\} = 0.11$, $\rho_{opt} = M\{\rho_{ad}\} = 1.1$. Figures 3, 4 shows that use of adaptive filtering allows keeping the optimal estimation accuracy for an infinite interval, and adaptive filtering accuracy is not very different from KF accuracy for this level of uncertainty. The 'real' RMS value is determined by averaging the estimation error squared (17) (18) using all samples. The 'calculated' RMS value is the square root of the mean value of the variances calculated by (17) (18). The coincidence of these values indicates the correctness of their calculation. The transition process for model parameter estimation is slower compared to the estimation of the bias (Fig. 5.6). Note that the similar problem of determining the PSDs of noise components can be solved using the Allan variance as in [16]. The results have shown that optimal estimation provides a 3-5 times better accuracy than for the Allan variance method.

This fact proves that the integrated problem of bias optimal estimation and model identification can be efficiently solved by nonlinear filtering methods using the bank of Kalman filters. It should be also noted that in designing the error model, both the problems of parameter estimation and structure identification prove important [21].

Conclusions

The paper establishes the relationship between Allan variance and variance of bias estimation error received by averaging. These variances are shown to coincide under some conditions. Thus, Allan variance can be used to estimate the minimum error variance of bias estimation by averaging and the corresponding averaging time, which is exactly important in sensor calibration. This relationship is illustrated by a model being a sum of white noise and random walk.

The paper discusses an approach improving the bias estimation accuracy with unknown sensor error model based on nonlinear filtering methods. It should be also noted that nonlinear filtering provides both estimation of error model parameters and identification of model structure.

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References

1. **Sveshnikov, A.A.,** Prikladnye zadachi teorii sluchainykh funktsii (Applied Problems of Theory of Random Functions)**,** Nauka, 1968.
2. **Bendat, J., Allan, G.P.** Random Data: analysis and measurement procedures, Wiley Interscience, New York (2nd ed 1986, 3rd ed. 2000, 4th ed. 2010)
3. **Prokhorov, S.A.,** Prikladnoi analiz sluchainykh protsessov (Applied Analysis of Applied Processes), Samara, 2007.
4. **Shevlyakov, G.L. and Lyubomischenko, N.S.,** Robust Estimation of Time Series Spectra: Brief Overview of Methods and Algorithms, Materialy XXIX konferentsii pamyati N.N. Ostryakova (Proceedings of the 29th Conference in Memory of N.N. Ostryakov), 2014.
5. **Sergienko, A.B.,** Tsifrovaya obrabotka signalov (Digital Signal Processing), St. Petersburg, 2002.
6. **Stepanov, O. A.,** Osnovy teorii otsenivaniya s prilozheniyami k zadacham obrabotki navigatsionnoi informatsii. Part 1. Vvedenie v teoriyu fil'tratsii (Fundamentals of the Estimation Theory with Applications to the Problems of Navigation Information Processing. Part 2. Introduction to Filtering Theory), St. Petersburg: CSRI Elektropribor, 2012**.**
7. **Allan, D.W.,** Statistics of Atomic Frequency Standards, Proc. of the IEEE, 54(2), 1966, pp. 221-230.
8. **Howe, D., Allan, D. and Barnes, J.**, Properties of Signal Sources and Measurement Methods, Proceedings of the 35th Annual Symposium on Frequency Control, 1981, pp. 464–469.
9. **Tehrani**, **M. M.** , Ring Laser Gyro Data Analysis with Cluster Sampling Technique, Proceedings of SPIE, vol. 412, 1983.
10. **Ng, L.C. and Pines, D.J.,** Characterization of Ring Laser Gyro Performance Using the Allan Variance Method, Journal of Guidance, Control, and Dynamics, Vol. 20, No. 1 (1997), pp. 211-214.
11. **Kucherkov, S.G. et al.,** Using Allan Variance in Characterization of MEMS Gyro**,** Giroskopyia i Navigatsiya, №2 (41), 2003, pp. 98-104.
12. **Siraya, T.N.,** Allan Variance as an Estimate of Measurement Error, Giroskopiya i Navigatsiya, 2010, №2(69), pp. 29– 36.
13. **Krobka, N.I.,** Differential Methods Of Identifying Gyro Noise Structure Gyroscopy And Navigation, 3, vol.2, 2011, hh. 126-137
14. **Stepanov, O.A., Toropov, A.B.** Comparison of Point Mass and Particle Filters in Map-aided Navigation // 17 St. Petersburg International Conf. on Integrated Navigation System, ICINS 2010 – Proceedings, pp. 381-384.
15. **Stepanov, O.A., Toropov, A.B.** Application of the Monte Carlo Methods with Partial Analytical Integration Techniques to the Problem of Navigation System Aiding // 20th Saint Petersburg International Conference on Integrated Navigation Systems, ICINS 2013 – Proceedings pp.298-301.
16. **Motorin, A.V. and Stepanov, O.A.,** Identification of Sensor Errors: Allan Variance vs Nonlinear Filtering, 21st St. Petersburg International Conference on Integrated Navigation Systems, 2014, pp. 123-128**.**
17. **Stepanov, O.A., Motorin, A.V., Vasil'ev, V.A., and Toropov, A.B.,** Using Nonlinear Filtering Methods in Designing Models for Sensor and Map Errors, Materialy XXIX konferentsii pamyati N.N. Ostryakova (Proceedings of the 29th Conference in Memory of N.N. Ostryakov), 2014.
18. **IEEE Std 528-2001**, IEEE Standard for Inertial Sensor Terminology, IEEE Aerospace and Electronic Systems Society, 2001.
19. **IEEE std. 1554-2005**, IEEE Recommended Practice for Inertial Sensor Test Equipment, Instrumentation, Data Acquisition, and Analysis. IEEE Aerospace and Electronic Systems Society, 2005.
20. **Stepanov, O.A. and Toropov, A.B.,** Using Sequential Monte-Carlo Methods and Analytical Integration in Navigation Data Processing, XII Vserossiiskoe soveshchanie po problemam upravleniya (12th Russian Meeting on Control Problems), Russia, Moscow, 16-19 June 2014.
21. **Motorin, A.V., Toropov, A.B., and Stepanov, O.A.,** Applying Multiple Model Filtering to the Estimation of Sensor Error Model, Materialy XVII konferentsii molodykh uchenykh "Navigatsiya i Upravlenie Dvizheniem" (Proceedings of the 17th Conference of Young Scientists "Navigation and Motion Control", 2015.
22. **Bucy, R. S.** Digital synthesis of non-linear filters./ R. S. Bucy and K. D. Senne //Automatica 7(3), 1971, pp.287-298, 1971.
23. **Allan, D. W.** Historicity, Strengths, and Weaknesses of Allan Variances and Their General Applications // Methods for Navigation Sensor erformance Determination, SPb, CSRI Elektropribor, 2015, pp. 509-526.