ISSN 2075-1087, Gyroscopy and Navigation, 2025, Vol. 16, No.2, pp. 131–139
Original Russian Text A.A. Golovan, I.M. Markov, I.V. Solov'ev, and M.A. Shatskii published in Giroskopiya i Navigatsiya, 2025, Vol. 33, No. 2, pp. 32–47

Decomposed Spacecraft Motion Estimation Algorithm Using GNSS Data

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Received: April 21, 2025; reviewed: June 28, 2025; accepted: July 9, 2025

Abstract: The article discusses the problem of estimating the spacecraft trajectory parameters based on data provided by GNSS (Global Navigation Satellite System) receiver and onboard spacecraft motion prediction algorithms. The traditional approach to solving the mentioned problem relies on Kalman filtering techniques for estimating the spacecraft's coordinates and velocity. In this case the navigation algorithms in the onboard computer must meet the technical requirements for maximum reduction in computational load. The paper proposes a method of decomposition for onboard integration algorithm using three parallel second-order filters, which significantly reduces the computational load of the algorithm while maintaining the accuracy.

Keywords: navigation, decomposed algorithm, spacecraft, GNSS-based navigation.

INTRODUCTION

Spacecraft autonomous navigation is one of key problems providing the spacecraft payload operation. Currently this problem is solved based on the data from the onboard spacecraft motion prediction algorithm and GNSS-derived data. Different approaches to this problem are covered in some Russian and international publications [1–5].

The problem of estimating the motion parameters of the vehicle center of mass (coordinates and components of the velocity vector) is solved on board the spacecraft using two information sources:

1) motion prediction with unlimited growth of errors with time;

2) GNSS measurements with the errors modeled by the stationary random process.

To finally estimate the motion parameters of the spacecraft center of mass, these data are fused to generate the integrated solution, which is the mean-square optimal estimate.

Further we explain the problem statement of fusing the data from onboard spacecraft motion prediction algorithm and GNSS data in the general form, similar to the description of strapdown inertial navigation system aiding in [6].

Let the behavior of a dynamic object – the spacecraft – be described by the equations

$$\dot{x} = f(x,u),\ x(t_0) = x_0,$$

where x is the state vector, u is the data on the external forces acting on the spacecraft.

The current state vector x is determined using the following data:

$$x_0' = x_0 + \delta x_0,\ u' = u + \delta u,$$

where δx_0 is the error of the initial state vector, δu is the error in modeling u.

The system defining the state vector x is described with the equations

$$\dot{x}' = f(x',u'), x'(t_0) = x_0',$$

which are further traditionally referred to as the mechanization equations.

The error equation for the value

$$\delta x = x - x'$$

is given as follows in the linearized form:

$$\delta\dot{x} = F\delta x + q,\ F = \frac{\partial f(x',u')}{\partial x'},$$

$$q = \frac{\partial f(x',u')}{\partial u'}\delta u.$$

GNSS receiver provides the measurements z of the vector x:

$$z = x + \zeta,$$

where ζ is the error in GNSS measurements.

The linearized measurement is generated:

$$\delta z = z - x' = H\delta x + \zeta. \quad (1)$$

The estimator structure using Kalman filtering is

$$\delta\dot{x} = F\delta x + K\left(\delta z - H\delta x\right), \quad (2)$$

where K is the gain selected according to the Kalman procedure.

The estimate $x^{(+)}$ of the state vector x is determined as

$$x^{(+)} = x' + \delta x^{(+)}.$$

In practice, relevant discrete relations are used.

The available methods to solve the considered problem are based on the Kalman filter to estimate three coordinates and three components of the velocity vector [7]. This reduces the effect of noise components on the estimation accuracy, simplifies the onboard motion prediction model, provides monitoring of GNSS measurements and detecting the anomalous measurements.

Numerical implementation of the algorithm with the traditional Kalman filter in onboard computer requires considerable computational loads. Further we describe a method to decompose the full-size sixth-order filter into three parallel second-order filters, which significantly reduces the computational loads with no degradation in the navigation accuracy. A similar method has been earlier used by the authors for spacecraft attitude determination in stellar aiding mode [8].

In [8], the spacecraft attitude parameters (parameters of its angular motion as a solid body with respect to its center of mass) are estimated by the star sensor measurements. The star sensor generates the quaternion of attitude of its body frame with respect to the inertial frame (IF). Here, a similar approach is applied to the spacecraft navigation problem, i.e., estimating the motion parameters of the center of mass (coordinates and components of the linear velocity vector) by GNSS-derived position and velocity. In space flight, these tasks are separated. The proposed algorithms can also be employed for the free flight navigation of upper stages of launch vehicles.

Note that the numerical implementation of the Kalman filter is important both for the reduction of computational load and stability of computations. Different approaches were suggested to reduce the computational burden during the filter implementation [9, 10]: simplifying the model of the dynamic system, reducing the dimensionality of the state vector, decomposing the state vector into several subvectors with lower dimensionality, sequential processing of measurements, using suboptimal filters. Currently, the Kalman filter U-D square root modification is effectively used [11], which is stable to machine rounding, includes no square root extraction or matrix inversion operations, and uses sequential scalar measurements updates. This modification of U-D filter has been implemented in the navigation algorithms described below.

It should be noted that computational loads of the Kalman filter nonlinearly depend on its dimensionality: the number of memory cells is proportional to the squared dimensionality of the state vector, and the number of addition and multiplication operations is proportional to the cubed dimensionality. Therefore, using a number of lower dimensionality filters, which, however, provide the required accuracy – if it is possible in a certain estimation problem – will be justified, since it reduces the computational load on the onboard computer.

When designing the decomposed navigation algorithms, we used the techniques for decomposing the estimation problems by the measurement vector components. For the details of these techniques, the reader is referred, for example, to [6].

An important aspect of the considered problem is the comparative analysis of accuracies of the proposed decomposed algorithms and the full-scale optimal algorithm.

The main objective of this paper is to design a decomposed algorithm for integrated data processing.

The paper is structured as follows. At first, mechanization equations of the studied problem are provided. Then, spacecraft navigation problem is stated as a problem of aiding (updating) the motion prediction results by GNSS measurements. Based on its solution, the optimal data fusion algorithm is generated, which serves as a model for the data fusion algorithm. Further, the model is used to design the decomposed estimation algorithms. The results from covariance analysis of accuracy of the proposed algorithms and the optimal one are provided. Some simulation results are given by the example of estimating motion parameters during free flight.

MECHANIZATION EQUATIONS OF THE NAVIGATION PROBLEM AS A PROBLEM OF ESTIMATING THE STATE VECTOR BY GNSS MEASUREMENTS

We formulate the problem of estimating the state vector

$$x(t) = \begin{pmatrix} r(t) \\ v(t) \end{pmatrix}$$

by discrete GNSS measurements

$$z_k = \begin{pmatrix} z_{r,k} \\ z_{v,k} \end{pmatrix} = x(t_k) + \zeta_k = x(t_k) + \begin{pmatrix} \zeta_{r,k} \\ \zeta_{v,k} \end{pmatrix}$$

obtained at times $t_k, k = 0, 1, 2, \ldots$.

Here,

$$r(t) = \begin{pmatrix} r_1(t) & r_2(t) & r_3(t) \end{pmatrix}^T$$

are the spacecraft coordinates in IF;

$$v(t) = \begin{pmatrix} v_1(t) & v_2(t) & v_3(t) \end{pmatrix}^T$$

is the spacecraft absolute velocity vector in IF;

$$z_{r,k} = r(t_k) + \zeta_{r,k},\ z_{v,k} = v(t_k) + \zeta_{v,k}$$

are GNSS position and velocity measurements; $\zeta_{r,k}, \zeta_{v,k}$ are the random errors of GNSS measurements, being white discrete Gaussian noises with the a priori characteristics:

$$M\{\zeta_{r,k}\} = 0_{3\times1},\ M\{\zeta_{r,i}\zeta_{r,k}^T\} = \sigma_r^2 I_{3\times3}\delta_{ik};$$

$$M\{\zeta_{v,k}\} = 0_{3\times1},\ M\{\zeta_{v,i}\zeta_{v,k}^T\} = \sigma_v^2 I_{3\times3}\delta_{ik};$$

$$M\{\zeta_k\zeta_k^T\} = R = \begin{pmatrix} \sigma_r^2 I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \sigma_v^2 I_{3\times3} \end{pmatrix}.$$

Here $M\{\cdots\}$ is the symbol of mathematical expectation, I is the identity matrix, δ_{ik} is the Kronecker symbol.

Let the IF, where aiding GNSS data are provided, be defined as follows [12]: the origin is at the Earth's center of mass, axis z is directed to the Celestial North Pole, axis x lies in the equatorial plane and is directed to the vernal equinox, axis y complements the system to the right-handed frame. The Celestial Pole and the vernal equinox correspond to the standard epoch 2000.0 (JD2000.0).

Synchronization of GNSS measurements and motion prediction solutions is not covered in the paper to avoid abundant technical details, the possible timing skew is considered to be relatively small.

Mechanization equations of spacecraft motion are

$$\dot{r}(t) = v(t), \tag{3}$$

$$\dot{v}(t) = g(r) + q_w(t). \tag{4}$$

Here, $g(r)$ is the vector of gravity force in inertial frame, $q_w(t)$ are the random disturbances (forces) acting on the spacecraft.

At the considered altitudes of the orbital motion (geostationary, high elliptical orbits), the atmospheric drag effect on the spacecraft motion is relatively low, and it can be modeled with the noise component of the motion model errors. With this assumption, the vector $q_w(t)$ is further modeled with white noise with a priori known characteristics:

$$M\{q_w(t)\} = 0_{3\times1},$$

$$M\{q_w(t)q_w^T(t+\tau)\} = \sigma_w^2 I_{3\times3}\delta(\tau) = Q_w\delta(\tau),$$

$$Q_w \equiv \sigma_w^2 I_{3\times3}.$$

The motion equations (3), (4) can be jointly written as

$$\dot{x}(t) = f(x(t)) + q(t),$$

where

$$f = \begin{pmatrix} x_4 & x_5 & x_6 & -\mu\frac{x_1}{|r|^3} & -\mu\frac{x_2}{|r|^3} & -\mu\frac{x_3}{|r|^3} \end{pmatrix}^T,$$

$$q = \begin{pmatrix} 0 & 0 & 0 & q_{w1} & q_{w2} & q_{w3} \end{pmatrix}^T.$$

The equations can also be written in matrix form:

$$\dot{x}(t) = \begin{pmatrix} 0_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{pmatrix} x(t) + \begin{pmatrix} 0_{3\times1} \\ g(r)_{3\times1} \end{pmatrix} + \begin{pmatrix} 0_{3\times1} \\ q_w(t)_{3\times1} \end{pmatrix}.$$

The model equations of the motion prediction implemented in the onboard computer are given by

$$\dot{r}'(t) = v'(t),$$

$$\dot{v}'(t) = g'(r'),$$

where $r'(t)$ are the inertial coordinates of the spacecraft, $v'(t)$ is the vector of spacecraft absolute velocity in IF, $g'(r')$ is the gravity force vector in IF calculated using the known model.

For definiteness, we shall use the model of the central gravitational field. Then we have

$$g'(r') = \begin{pmatrix} -\mu\frac{r_1'}{|r'|^3} & -\mu\frac{r_2'}{|r'|^3} & -\mu\frac{r_3'}{|r'|^3} \end{pmatrix}^T,$$

$$|r'| = \sqrt{(r_1')^2 + (r_2')^2 + (r_3')^2},$$

where $\mu = 398600.4415 \cdot 10^9$ m^3/s^2 is the Earth's gravitation constant.

We introduce the motion prediction error vector

$$\delta x(t) = x(t) - x'(t) = \begin{pmatrix} \delta r(t) \\ \delta v(t) \end{pmatrix},\ x'(t) = \begin{pmatrix} r'(t) \\ v'(t) \end{pmatrix}$$

and the vector of differences between GNSS measurements and model parameters

$$\delta z_k = x'(t_k) - z_k = \begin{pmatrix} z_{r,k} - r'(t_k) \\ z_{v,k} - v'(t_k) \end{pmatrix} = \begin{pmatrix} \delta z_{r,k} \\ \delta z_{v,k} \end{pmatrix} = \begin{pmatrix} \delta r(t_k) + \zeta_{r,k} \\ \delta v(t_k) + \zeta_{v,k} \end{pmatrix}.$$

Motion prediction error equations are

$$\delta \dot{x}(t) = \begin{pmatrix} \delta \dot{r}(t) \\ \delta \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0_{3\times3} & I_{3\times3} \\ G_{3\times3}(r) & 0_{3\times3} \end{pmatrix} \delta x(t) + \begin{pmatrix} 0_{3\times1} \\ I_{3\times1} \end{pmatrix} q_w(t) = F(t)\delta x(t) + Bq_w(t),$$

$$F(t) = \begin{pmatrix} 0_{3\times3} & I_{3\times3} \\ G_{3\times3}(r) & 0_{3\times3} \end{pmatrix},\ B = \begin{pmatrix} 0_{3\times1} \\ I_{3\times1} \end{pmatrix},$$

where $G(r) = g_{ij}(r) = \partial g_i(r) / \partial r_j$, $i, j = 1,2,3$ is the gravity tensor.

For the central gravitational field model, we have

$$G(r) = -\frac{\mu}{|r|^3} I_{3\times3} + \frac{3\mu}{|r|^5} \begin{pmatrix} r_1^2 & r_1 r_2 & r_1 r_3 \\ r_1 r_2 & r_2^2 & r_2 r_3 \\ r_1 r_3 & r_2 r_3 & r_3^2 \end{pmatrix}.$$

The measurement equations are

$$\delta z_k = \begin{pmatrix} I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} \end{pmatrix} \delta x(t_k) + \begin{pmatrix} \zeta_{r,k} \\ \zeta_{v,k} \end{pmatrix} = H\delta x(t_k) + \zeta_k,$$

$$H = I_{6\times6},\ \zeta_k = \begin{pmatrix} \zeta_{r,k} \\ \zeta_{v,k} \end{pmatrix}.$$

ESTIMATION PROBLEM: OPTIMAL KALMAN FILTER

Formulate the problem of estimating the state vector $x(t)$ by measurements z_k in continuous time:

$$\dot{x}(t) = f(x(t)) + q(t),$$
$$z_k = Hx(t_k) + \zeta_k.$$

To construct the extended Kalman filter, we linearize the dynamic system model in the vicinity of the predicted trajectory:

$$x(t) = \hat{x}(t) + \delta x(t),$$

where $\delta x(t)$ is the estimate error vector, $\hat{x}(t) = M\{x(t)\}$.

Then

$$\delta \dot{x}(t) = \left(\frac{\partial f}{\partial x}\right)_{x=\hat{x}} \delta x(t) + q(t) = F(t)\delta x(t) + q(t).$$

Hence, we get a linear problem of estimating the error vector $\delta x(t)$ by difference aiding measurements z_k according to (1), (2):

$$\delta \dot{x}(t) = F(t)\delta x(t) + Bq_w(t),$$
$$\delta z_k = H\delta x(t_k) + \zeta_k.$$

The corresponding linear estimation problem in discrete time is given by:

$$\delta x_k = \Phi_{k|k-1}\delta x_{k-1} + Bq_{w,k},$$
$$\delta z_k = H\delta x_k + \zeta_k,$$

where $\Phi_{k|k-1}$ is the transition matrix, matrices B, H are identical to the similar matrices for continuous model, $q_{w,k}$ is the discrete white noise equivalent to noise q_w with intensity matrix Q_k.

Transition matrix$\Phi_{k|k-1}$ and the system noise matrix Q_k are given by

$$\Phi_{k|k-1} \approx I + F(t_{k-1})\Delta t = \begin{pmatrix} I_{3\times3} & I_{3\times3}\Delta t \\ G(r')\Delta t & I_{3\times3} \end{pmatrix},\ \Delta t = t_k - t_{k-1},\ Q_k = \int_{t_{k-1}}^{t_k} \Phi_{k|k-1}(t_k, \tau) B q_w B^T \Phi_{k|k-1}^T(t_k, \tau) d\tau =$$

$$= \int_{t_{k-1}}^{t_k} \begin{pmatrix} I_{3\times3} & I_{3\times3}(t_k - \tau) \\ G(t_k - \tau) & I_{3\times3} \end{pmatrix} \begin{pmatrix} 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & Q_w \end{pmatrix} \begin{pmatrix} I_{3\times3} & G(t_k - \tau) \\ I_{3\times3}(t_k - \tau) & I_{3\times3} \end{pmatrix} d\tau = \int_{t_{k-1}}^{t_k} \begin{pmatrix} Q_w (t_k - \tau)^2 & Q_w (t_k - \tau) \\ Q_w (t_k - \tau) & Q_w \end{pmatrix} d\tau =$$

$$= \begin{pmatrix} I_{3\times3}\Delta t^2 / 3 & I_{3\times3}\Delta t / 2 \\ I_{3\times3}\Delta t / 2 & I_{3\times3} \end{pmatrix} \sigma_w^2 \Delta t.$$

Introduce the following denotations:

$\delta x_k^{(-)}$, $\delta x_k^{(+)}$ are a priori and a posteriori estimates of the error vector δx_k;

$\Delta x_k^{(-)} = \delta x_k - \delta x_k^{(-)}$, $\Delta x_k^{(+)} = \delta x_k - \delta x_k^{(+)}$ are a priori and a posteriori estimation errors;

$P_{\Delta x,k}^{(-)} \equiv P_k^{(-)} = M\left\{\Delta x_k^{(-)} \Delta x_k^{(-)T}\right\}$, $P_{\Delta x,k}^{(+)} \equiv P_k^{(+)} = M\left\{\Delta x_k^{(+)} \Delta x_k^{(+)T}\right\}$ are the covariance matrices of a priori and a posteriori estimation errors, respectively.

The optimal estimation algorithm is described in (5) – (13) [10].

A priori estimates (prediction):

$$x_k^{(-)} = f\left(x_{k-1}^{(+)}\right), \quad (5)$$

$$\delta x_k^{(-)} = 0 \quad (6)$$

(mathematical expectation of a priori estimation error is identically equal to zero),

$$P_k^{(-)} = \Phi_{k|k-1} P_{k-1}^{(+)} \Phi_{k|k-1}^T + Q_k. \quad (7)$$

A posteriori estimates (measurement update):

$$\delta z_k = z_k - x_k^{(-)}, \quad (8)$$

$$K_k = P_k^{(-)} H^T \left(H P_k^{(-)} H^T + R\right)^{-1}, \quad (9)$$

$$\delta x_k^{(+)} = \delta x_k^{(-)} + K_k\left(\delta z_k - H \delta x_k^{(-)}\right) = K_k \delta z_k, \quad (10)$$

$$P_k^{(+)} = \left(I - K_k H\right) P_k^{(-)}, \quad (11)$$

$$x_k^{(+)} = x_k^{(-)} + \delta x_k^{(+)}. \quad (12)$$

Initial conditions

$$x_0^{(+)} = z_0,\ \delta x_0^{(+)} = 0,\ P_0^{(+)} = P_0. \quad (13)$$

ESTIMATION PROBLEM: DECOMPOSED ALGORITHM

Introduce three error vectors

$$\delta x^{(1)} = \begin{pmatrix} \delta r_1 \\ \delta v_1 \end{pmatrix},\ \delta x^{(2)} = \begin{pmatrix} \delta r_2 \\ \delta v_2 \end{pmatrix},\ \delta x^{(3)} = \begin{pmatrix} \delta r_3 \\ \delta v_3 \end{pmatrix}$$

and three vectors of aiding difference measurements:

$$\delta z^{(1)} = \begin{pmatrix} \delta r_1 \\ \delta v_1 \end{pmatrix} + \begin{pmatrix} \zeta_{r,1} \\ \zeta_{v,1} \end{pmatrix},\ \delta z^{(2)} = \begin{pmatrix} \delta r_2 \\ \delta v_2 \end{pmatrix} + \begin{pmatrix} \zeta_{r,2} \\ \zeta_{v,2} \end{pmatrix},$$

$$\delta z^{(3)} = \begin{pmatrix} \delta r_3 \\ \delta v_3 \end{pmatrix} + \begin{pmatrix} \zeta_{r,3} \\ \zeta_{v,3} \end{pmatrix}.$$

Motion prediction error equations in discrete form are given by

$$\delta r_{i,k} = \delta r_{i,k-1} + \delta v_{i,k-1} \Delta t,$$
$$\delta v_{i,k} = \delta v_{i,k-1} + g_{ii,k-1} \delta r_{i,k-1} \Delta t + u_{i,k-1},$$

where parameters $u_{j,k}$ are

$$u_{1,k-1} = \left(g_{12,k-1} \delta r_{2,k-1} + g_{13,k-1} \delta r_{3,k-1}\right) \Delta t,$$
$$u_{2,k-1} = \left(g_{12,k-1} \delta r_{1,k-1} + g_{23,k-1} \delta r_{3,k-1}\right) \Delta t,$$
$$u_{3,k-1} = \left(g_{13,k} \delta r_{1,k-1} + g_{23,k-1} \delta r_{2,k-1}\right) \Delta t.$$

Further in decomposed aiding algorithms, particularly, at the extrapolation step, $u_{j,k}$ are considered to be the known inputs generated by the estimates of the relevant decomposed filters:

$$\tilde{u}_{1,k-1} = \left(g_{12,k-1} \delta r_{2,k-1}^{(+)} + g_{13,k-1} \delta r_{3,k-1}^{(+)}\right) \Delta t,$$
$$\tilde{u}_{2,k-1} = \left(g_{12,k-1} \delta r_{1,k-1}^{(+)} + g_{23,k-1} \delta r_{3,k-1}^{(+)}\right) \Delta t,$$
$$\tilde{u}_{3,k-1} = \left(g_{13,k-1} \delta r_{1,k-1}^{(+)} + g_{23,k-1} \delta r_{2,k-1}^{(+)}\right) \Delta t.$$

Transition matrices of decomposed filters are

$$\Phi_{k|k-1}^{(i)} = \begin{pmatrix} 1 & \Delta t \\ g_{ii,k-1} \Delta t & 1 \end{pmatrix}, i = 1, 2, 3.$$

Parameter $u_{j,k}$ is introduced in order to decompose the sixth-order filter into three independent second-order filters operating sequentially one after another. In each second-order filter, $u_{j,k}$ is a known value that can be treated as a specified input directly affecting the estimated parameters.

The decomposed estimation algorithm is described with (14) – (43).

A priori estimates (prediction):

$$x_k^{(-)} = f\left(x_{k-1}^{(+)}\right), \quad (14)$$

$$\tilde{u}_{1,k-1} = \left(g_{12,k-1} \delta x_{k-1}^{(2)(+)} + g_{13,k-1} \delta x_{k-1}^{(3)(+)}\right) \Delta t, \quad (15)$$

$$\tilde{u}_{2,k-1} = \left(g_{12,k-1} \delta x_{k-1}^{(1)(+)} + g_{23,k-1} \delta x_{k-1}^{(3)(+)}\right) \Delta t, \quad (16)$$

$$\tilde{u}_{3,k-1} = \left(g_{13,k-1} \delta x_{k-1}^{(1)(+)} + g_{23,k-1} \delta x_{k-1}^{(2)(+)}\right) \Delta t, \quad (17)$$

$$\delta x_k^{(1)(-)} = \begin{pmatrix} 0 \\ \tilde{u}_{1,k-1} \end{pmatrix}, \quad (18)$$

$$\delta x_k^{(2)(-)} = \begin{pmatrix} 0 \\ \tilde{u}_{2,k-1} \end{pmatrix}, \quad (19)$$

$$\delta x_k^{(3)(-)} = \begin{pmatrix} 0 \\ \tilde{u}_{3,k-1} \end{pmatrix}, \quad (20)$$

$$P_k^{(1)(-)} = \Phi_{k|k-1}^{(1)} P_{k-1}^{(1)(+)} \Phi_{k|k-1}^{(1)T} + Q_k^{(1)}, \quad (21)$$

$$P_k^{(2)(-)} = \Phi_{k|k-1}^{(2)} P_{k-1}^{(2)(+)} \Phi_{k|k-1}^{(2)T} + Q_k^{(2)}, \quad (22)$$

$$P_k^{(3)(-)} = \Phi_{k|k-1}^{(3)} P_{k-1}^{(3)(+)} \Phi_{k|k-1}^{(3)T} + Q_k^{(3)}, \quad (23)$$

$$Q_k^{(1)} = \begin{pmatrix} Q_{11,k} & Q_{14,k} \\ Q_{14,k} & Q_{44,k} \end{pmatrix}, \; Q_k^{(2)} = \begin{pmatrix} Q_{22,k} & Q_{25,k} \\ Q_{25,k} & Q_{55,k} \end{pmatrix}, \quad Q_k^{(3)} = \begin{pmatrix} Q_{33,k} & Q_{36,k} \\ Q_{36,k} & Q_{66,k} \end{pmatrix}. \quad (24)$$

A posteriori estimates (measurement update):

$$\delta z_k^{(1)} = z_k^{(1)} - x_k^{(1)(-)}, \quad (25)$$

$$\delta z_k^{(2)} = z_k^{(2)} - x_k^{(2)(-)}, \quad (26)$$

$$\delta z_k^{(3)} = z_k^{(3)} - x_k^{(3)(-)}, \quad (27)$$

$$K_k^{(1)} = P_k^{(1)(-)} H^T \left(H P_k^{(1)(-)} H^T + R^{(1)} \right)^{-1}, \quad (28)$$

$$K_k^{(2)} = P_k^{(2)(-)} H^T \left(H P_k^{(2)(-)} H^T + R^{(2)} \right)^{-1}, \quad (29)$$

$$K_k^{(3)} = P_k^{(3)(-)} H^T \left(H P_k^{(3)(-)} H^T + R^{(3)} \right)^{-1}, \quad (30)$$

$$R_k^{(1)} = \begin{pmatrix} R_{11,k} & R_{14,k} \\ R_{14,k} & R_{44,k} \end{pmatrix}, \; R_k^{(2)} = \begin{pmatrix} R_{22,k} & R_{25,k} \\ R_{25,k} & R_{55,k} \end{pmatrix}, \quad R_k^{(3)} = \begin{pmatrix} R_{33,k} & R_{36,k} \\ R_{36,k} & R_{66,k} \end{pmatrix}, \quad (31)$$

$$\delta x_k^{(1)(+)} = \delta x_k^{(1)(-)} + K_k^{(1)} \left(\delta z_k^{(1)} - H \delta x_k^{(1)(-)} \right), \quad (32)$$

$$\delta x_k^{(2)(+)} = \delta x_k^{(2)(-)} + K_k^{(2)} \left(\delta z_k^{(2)} - H \delta x_k^{(2)(-)} \right), \quad (33)$$

$$\delta x_k^{(3)(+)} = \delta x_k^{(3)(-)} + K_k^{(3)} \left(\delta z_k^{(3)} - H \delta x_k^{(3)(-)} \right), \quad (34)$$

$$P_k^{(1)(+)} = \left(I - K_k^{(1)} H \right) P_k^{(1)(-)}, \quad (35)$$

$$P_k^{(2)(+)} = \left(I - K_k^{(2)} H \right) P_k^{(2)(-)}, \quad (36)$$

$$P_k^{(3)(+)} = \left(I - K_k^{(3)} H \right) P_k^{(3)(-)}. \quad (37)$$

Updating the state vector:

$$x_{1,k}^{(+)} = x_{1,k}^{(-)} + \delta x_{1,k}^{(1)(+)}, \quad (38)$$

$$x_{2,k}^{(+)} = x_{2,k}^{(-)} + \delta x_{1,k}^{(2)(+)}, \quad (39)$$

$$x_{3,k}^{(+)} = x_{3,k}^{(-)} + \delta x_{1,k}^{(3)(+)}, \quad (40)$$

$$x_{4,k}^{(+)} = x_{4,k}^{(-)} + \delta x_{2,k}^{(1)(+)} + \tilde{u}_{1,k-1}, \quad (41)$$

$$x_{5,k}^{(+)} = x_{5,k}^{(-)} + \delta x_{2,k}^{(2)(+)} + \tilde{u}_{2,k-1}, \quad (42)$$

$$x_{6,k}^{(+)} = x_{6,k}^{(-)} + \delta x_{2,k}^{(3)(+)} + \tilde{u}_{3,k-1}. \quad (43)$$

COVARIANCE ACCURACY ANALYSIS OF DECOMPOSED ALGORITHM

The composite state vector of the decomposed estimator is generated from the estimates of decomposed filters:

$$\delta \tilde{x}_{6\times 1} = \begin{pmatrix} \delta x_1^{(1)} & \delta x_1^{(2)} & \delta x_1^{(3)} & \delta x_2^{(1)} & \delta x_2^{(2)} & \delta x_2^{(3)} \end{pmatrix}^T.$$

Introduce a priori and a posteriori errors of decomposed algorithm estimates

$$\Delta \tilde{x}_k^{(-)} = \delta x_k - \delta \tilde{x}_k^{(-)}, \quad \Delta \tilde{x}_k^{(+)} = \delta x_k - \delta \tilde{x}_k^{(+)}$$

and relevant covariance matrices

$$\tilde{P}_{\Delta\tilde{x},k}^{(-)} \equiv \tilde{P}_k^{(-)} = M\left\{ \Delta \tilde{x}_k^{(-)} \Delta \tilde{x}_k^{(-)T} \right\},$$

$$\tilde{P}_{\Delta\tilde{x},k}^{(+)} \equiv \tilde{P}_k^{(+)} = M\left\{ \Delta \tilde{x}_k^{(+)} \Delta \tilde{x}_k^{(+)T} \right\}.$$

The decomposed algorithm differs from the optimal one only in the selection of the composite gain matrix $\tilde{K}_k$ composed of the gain factors generated by the decomposed filters:

$$\tilde{K}_k = \begin{pmatrix} k_{11}^{(1)} & 0 & 0 & k_{12}^{(1)} & 0 & 0 \\ 0 & k_{11}^{(2)} & 0 & 0 & k_{12}^{(2)} & 0 \\ 0 & 0 & k_{11}^{(3)} & 0 & 0 & k_{12}^{(3)} \\ k_{21}^{(1)} & 0 & 0 & k_{22}^{(1)} & 0 & 0 \\ 0 & k_{21}^{(2)} & 0 & 0 & k_{22}^{(2)} & 0 \\ 0 & 0 & k_{21}^{(3)} & 0 & 0 & k_{22}^{(3)} \end{pmatrix}.$$

Here $k_{11}^{(i)}, k_{12}^{(i)}, k_{21}^{(i)}, k_{22}^{(i)}, i = 1, 2, 3$ are the gains of three decomposed second-order filters.

In the composite decomposed algorithm the parameters of inputs $u_{i,k}$ are included in the transition matrix $\Phi_{k|k-1}$, so it remains unchanged.

The decomposed algorithm in composite form can be described with (44) – (48).

A priori estimates:

$$\delta \tilde{x}_k^{(-)} = 0, \quad (44)$$

$$\tilde{P}_k^{(-)} = \Phi_{k|k-1} \tilde{P}_{k-1}^{(+)} \Phi_{k|k-1}^T + Q_k. \quad (45)$$

A posteriori estimates:

$$\delta \tilde{x}_k^{(+)} = \delta \tilde{x}_k^{(-)} + \tilde{K}_k \left(\delta z_k - H \delta \tilde{x}_k^{(-)} \right) = \tilde{K}_k \delta z_k, \quad (46)$$

$$\tilde{P}_k^{(+)} = \left(I - \tilde{K}_k H \right) \tilde{P}_k^{(-)} \left(I - \tilde{K}_k H \right)^T + \tilde{K}_k R \tilde{K}_k^T, \quad (47)$$

$$\delta\tilde{x}_0^{(+)} = 0, \ \tilde{P}_0^{(+)} = P_0. \quad (48)$$

Introduce a priori and a posteriori relative errors of the decomposed algorithm:

$$Dx_k^{(-)} = \Delta\tilde{x}_k^{(-)} - \Delta x_k^{(-)} = \delta\tilde{x}_k^{(-)} - \delta x_k^{(-)},$$
$$Dx_k^{(+)} = \Delta\tilde{x}_k^{(+)} - \Delta x_k^{(+)} = \delta\tilde{x}_k^{(+)} - \delta x_k^{(+)}.$$

They satisfy the equations (49)–(52):

$$Dx_k^{(-)} = 0, \quad (49)$$

$$Dx_k^{(+)} = \left(I - \tilde{K}_k H\right) Dx_k^{(-)} - \Delta K_k H \delta x_k^{(-)} + \Delta K_k \delta z_k = \Delta K_k \delta z_k, \quad (50)$$

$$\Delta K_k = \tilde{K}_k - K_k, \quad (51)$$

$$Dx_0^{(+)} = 0. \quad (52)$$

Introduce the cross covariance matrices for a priori and a posteriori errors of the optimal and decomposed algorithms:

$$P_{c,k}^{(-)} = M\left\{\Delta\tilde{x}_k^{(-)} \Delta x_k^{(-)T}\right\} = M\left\{\Delta x_k^{(-)} \Delta\tilde{x}_k^{(-)T}\right\},$$
$$P_{c,k}^{(+)} = M\left\{\Delta\tilde{x}_k^{(+)} \Delta x_k^{(+)T}\right\} = M\left\{\Delta x_k^{(+)} \Delta\tilde{x}_k^{(+)T}\right\},$$
$$P_{c,0}^{(-)} = P_{c,0}^{(+)} = 0.$$

They satisfy the equations (53)–(54):

$$P_{c,k}^{(-)} = \Phi_{k|k-1} P_{c,k-1}^{(+)} \Phi_{k|k-1}^T + Q_k, \quad (53)$$

$$P_{c,k}^{(+)} = \left(I - K_k H\right) P_{c,k}^{(-)} \left(I - \tilde{K}_k H\right)^T + K_k R \tilde{K}_k^T. \quad (54)$$

Since it follows from (9) that

$$K_k R = \left(I - K_k H\right) P_k^{(-)} H^T,$$

we have

$$P_{c,k}^{(+)} = P_k^{(+)} = \left(I - K_k H\right) P_k^{(-)}.$$

Introduce the composite vectors of a priori and a posteriori errors

$$Dy_k^{(-)} = \begin{pmatrix} Dx_k^{(-)} \\ \delta x_k^{(-)} \end{pmatrix}, \ Dy_k^{(+)} = \begin{pmatrix} Dx_k^{(+)} \\ \delta x_k^{(+)} \end{pmatrix}$$

and their covariance matrices:

$$P_{y,k}^{(\pm)} = M\left\{Dy_k^{(\pm)} Dy_k^{(\pm)T}\right\} = \begin{pmatrix} Dx_k^{(\pm)} \\ \delta x_k^{(\pm)} \end{pmatrix} \begin{pmatrix} Dx_k^{(\pm)T} & \delta x_k^{(\pm)T} \end{pmatrix} = \begin{pmatrix} P_{DD,k}^{(\pm)} & P_{D\delta,k}^{(\pm)} \\ P_{\delta D,k}^{(\pm)} & P_{\delta\delta,k}^{(\pm)} \end{pmatrix},$$
$$P_{y,0}^{(\pm)} = 0.$$

Covariance matrices of a priori estimates are given by

$$P_{DD,k}^{(-)} = \Phi_{k|k-1} P_{DD,k-1}^{(+)} \Phi_{k|k-1}^T, \quad (55)$$

$$P_{D\delta,k}^{(-)} = \Phi_{k|k-1} \underbrace{\left(P_{c,k-1}^{(+)} - P_{k-1}^{(+)}\right)}_{=0} \Phi_{k|k-1}^T + Q_k = Q_k,$$

$$P_{\delta\delta,k}^{(-)} = P_k^{(-)} = \Phi_{k|k-1} P_{k-1}^{(+)} \Phi_{k|k-1}^T + Q_k.$$

Covariance matrices of a posteriori estimates are

$$P_{DD,k}^{(+)} = \left(I - \tilde{K}_k H\right) P_{DD,k}^{(-)} \left(I - \tilde{K}_k H\right)^T + \Delta K_k \left(H P_k^{(-)} H^T + R\right) \Delta K_k^T - \underbrace{\left(I - \tilde{K}_k H\right)\left(P_{c,k}^{(-)} - P_k^{(-)}\right) H^T \Delta K_k^T - \Delta K_k H \left(P_{c,k}^{(-)} - P_k^{(-)}\right)\left(I - \tilde{K}_k H\right)^T}_{=0};$$

$$P_{D\delta,k}^{(+)} = \left(I - \tilde{K}_k H\right) \underbrace{\left(P_{c,k}^{(-)} - P_k^{(-)}\right)}_{=0} - \Delta K_k H P_k^{(-)};$$

$$P_{\delta\delta,k}^{(+)} = P_k^{(+)} = \left(I - K_k H\right) P_k^{(-)}.$$

Finally, we have the following equations for a posteriori errors:

$$P_{DD,k}^{(+)} = \left(I - \tilde{K}_k H\right) P_{DD,k}^{(-)} \left(I - \tilde{K}_k H\right)^T + \Delta K_k \left(H P_k^{(-)} H^T + R\right) \Delta K_k^T, \quad (56)$$

$$P_{D\delta,k}^{(+)} = -\Delta K_k H P_k^{(-)},$$

$$P_{\delta\delta,k}^{(+)} = P_k^{(+)} = \left(I - K_k H\right) P_k^{(-)}.$$

Equations (55) and (56) describe the time behavior of relative errors of decomposed algorithm as compared to the optimal one. Note that they were derived with no consideration for the features of the certain studied dynamic system, thus they characterize the accuracy degradation of any Kalman type algorithm, differing from the Kalman filter by the selection of gain matrix, as compared to the optimal algorithm.

Analysis of computational complexity (number of arithmetic operations) of optimal and decomposed algorithms demonstrates that software implementation of the decomposed algorithm requires about five times less multiplication and division operations.

SIMULATION RESULTS

Below we provide some simulation results for the decomposed algorithm used to estimate the motion parameters of launch vehicle upper stage during the free flight (with turned off main engine) 2858 s long.

When modeling the Earth's anomalous gravitational field, we accounted for its terms up to the eighth order in expansion in spherical harmonics. The onboard algorithm model considered only the central gravitational field. The following parameters of GNSS noise error model were taken: σ_r = 15 m, σ_v = 0.05 m/s. The GNSS data sampling rate is 1 Hz.

Figure 1 presents the x-axis GNSS errors, errors in coordinates and velocity vector components for the decomposed algorithm with RMS errors.

Figure 2 presents the differences between the errors in x-axis coordinates and component of the velocity vector for the decomposed and optimal algorithms and their RMS errors computed using (55), (56). The error variations are due to the upper stage rotations about the Earth during the free flight. The curves reveal accuracy degradation as compared to the optimal algorithm of the order of 1 mm (0.07 %) in coordinates and 10^{-2} mm/s (0.02 %) in velocity vector components, which is negligibly small.

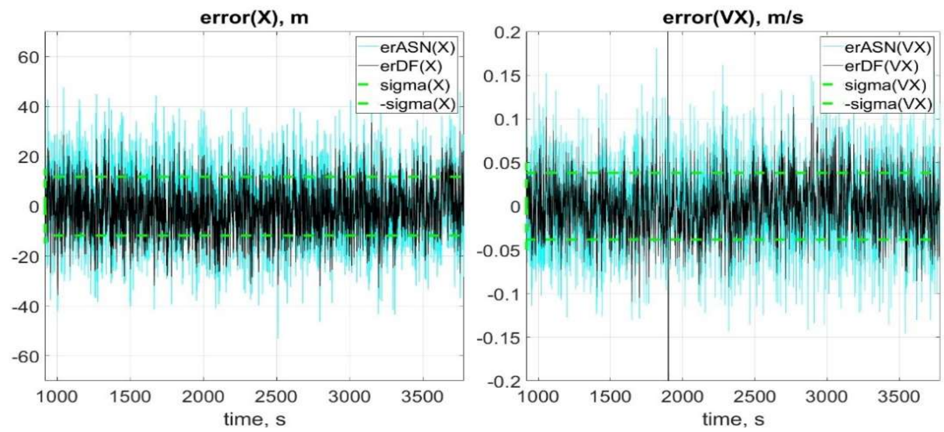

Fig. 1. Errors in *x*-axis coordinates and *x*-component of the velocity vector by GNSS data (blue lines, erASN(X), erASN(VX)) and by the decomposed algorithm (black lines, erDF(X), erDF(VX)) and RMS errors (± 1 RMS, sigma(X), sigma(VX)).

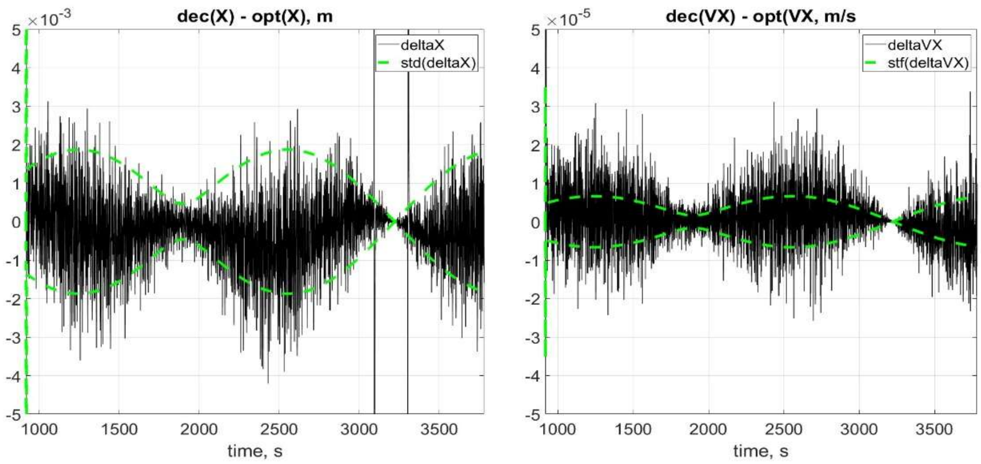

Fig. 2. Differences in errors in *x*-axis coordinates and *x*-component of the velocity vector for the decomposed and optimal algorithms (black lines deltaX, deltaVX) and their RMS errors (green lines std(deltaX), std(deltaVX)).

CONCLUSIONS

The paper focuses on the fusion of GNSS data and onboard motion prediction data for estimating the motion parameters of spacecraft or upper stage during the free flight. A method of decomposing the optimal sixth-order Kalman filter into three second-order filters has been proposed, which considerably reduces the algorithm computational complexity.

Formulas describing the time behavior of covariance matrices of relative errors for the decomposed algorithm as compared to the optimal one have been obtained. These formulas are true for a Kalman type algorithm differing from the Kalman filter by the selection of the gain matrix.

Covariance accuracy analysis of the decomposed algorithm as compared to the optimal estimator demonstrates that the loss in accuracy has an order of several mm in coordinates and several hundredth fractions of mm/s in velocity vector components (hundredth of percent). i.e., are negligibly small as compared to typical GNSS errors. These results prove the consistency of the proposed approach.

The algorithm is being implemented in the onboard software of advanced space systems.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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